Berlin Mathematics Research Center

MATH+

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Challenges and Solutions

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Illustration: Zyanya Santuario

1 Safe Christmas!

Author: Anouk Beursgens
Project: 4TU.AMI

Challenge

It is almost time to distribute the Christmas presents, but Santa Claus is nowhere to be found! All Christmas presents are stored in a safe, and Santa Claus appears to be the only person who knows how to open the safe. The safe is unique as it does not require a numerical code, but it opens once all lights are turned off or on. The elves are desperate and ask you to look and find a strategy to open the safe. You are the only one who can save Christmas! You decide to take a closer look at the safe. A large not rotatable disc with four light switches is on the door's exterior. Each switch can either be in an 'on' or 'off' position. Behind each switch is a light bulb that is either on or off, depending on the position of the switch controlling it. The bulbs are inside the safe; therefore, you cannot see them. The disc with the bulbs on the inside is rotatable.

The safe door opens once all lights are turned off, or all lights are turned on. You will use the switches on the safe door to open the safe. You can switch 1 or 2 switches at each try. After each try, the disc rotates an unknown number of quarter turns (given that the door did not open).

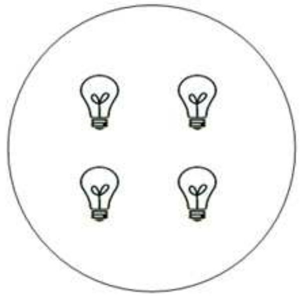


Figure 1: Interior of the lock of the safe.

Your task

It is up to you to design a strategy that always opens the safe, regardless of the current state of the lock. Give your strategy as a sequence of the actions you need to take in order to open the safe, where the possible actions are decoded in the following way:

- (1) : Pressing 1 button
- (D) : Pressing 2 diagonally adjacent buttons
- (A) : Pressing 2 horizontally or vertically adjacent buttons

If the safe is open once, by all lights being turned on or turned off, the game is over and Christmas is saved!

Hint: One way to start could be to write down all possible states where the lock can be. What happens when the disc with the bulbs rotates? Are there similar states? Consider how the given actions influence each state. Time to save Christmas!
Note: Explanation for understanding the switches: Each switch has two positions. When a switch is flipped, the state of the connected light bulb changes, regardless of the switch's previous position (i.e., it does not matter whether the switch is flipped from top to bottom or from bottom to top).

Possible Answers:

1. $(1) \rightarrow (A) \rightarrow (D)$
2. $(D) \rightarrow (A) \rightarrow (1) \rightarrow (D) \rightarrow (A)$
3. $(A) \rightarrow (D) \rightarrow (1) \rightarrow (A) \rightarrow (D)$
4. $(D) \rightarrow (A) \rightarrow (A) \rightarrow (1) \rightarrow (D) \rightarrow (A) \rightarrow (A)$
5. $(D) \rightarrow (A) \rightarrow (1)$
6. $(D) \rightarrow (A) \rightarrow (D) \rightarrow (1) \rightarrow (D) \rightarrow (A) \rightarrow (D)$
7. $(1) \rightarrow (A) \rightarrow (D) \rightarrow (1) \rightarrow (A) \rightarrow (D)$
8. $(A) \rightarrow (1) \rightarrow (A) \rightarrow (D)$
9. $(1) \rightarrow (D) \rightarrow (A)$
10. $(A) \rightarrow (1) \rightarrow (A) \rightarrow (D) \rightarrow (A) \rightarrow (1) \rightarrow (D)$

Solution

The correct answer is: 6.

As suggested, we start by writing down all possible states. At first, you might think of $2^4 = 16$ states, as each of the four lights is either on or off. However, since the disc with the light bulbs is rotating, the only meaningful information about the state is whether there are zero lights on (safe open), one light on, two diagonal lights on, or two adjacent lights on. Note that having three or four lights on is equivalent to having one or zero lights on, because the safe opens for both zero and four lights on.
For each state of the safe, we determine the effect of actions (1), (D), (A) and indicate this by an arrow (directed edge) from the state to the resulting state and label it by the action. This yields the directed labeled graph as in Figure 2.

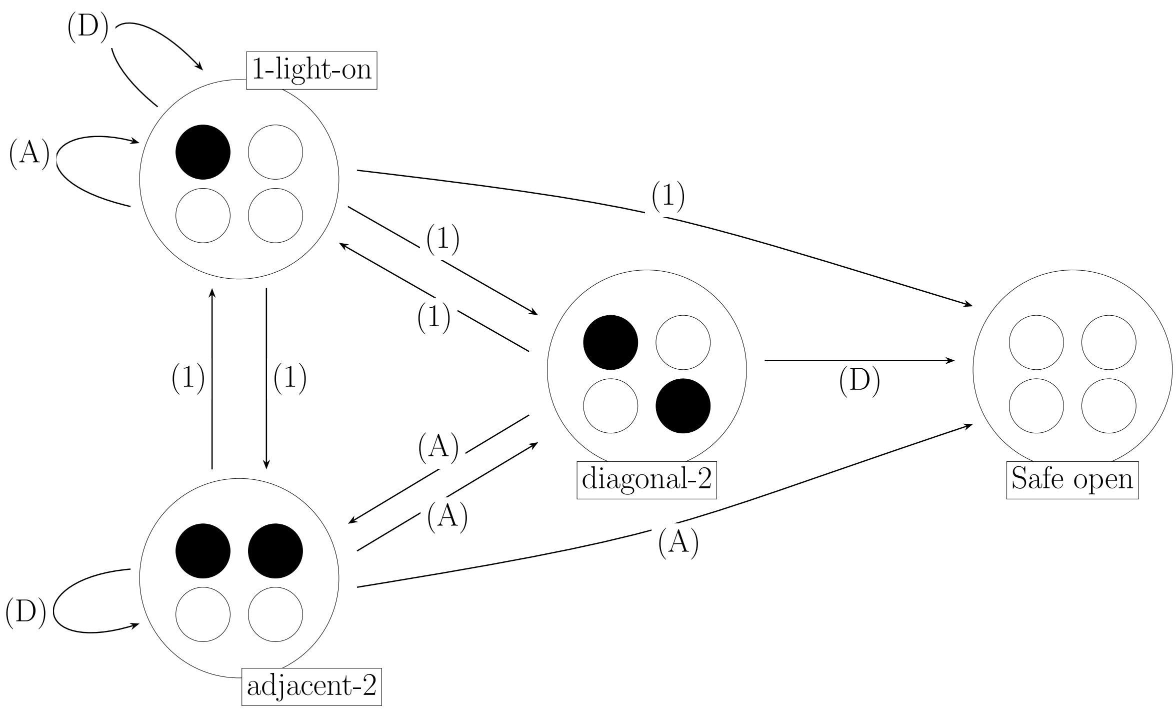


Figure 2: Graphical representation of the states and actions

The challenge remains to find a strategy (sequence of actions) that leads to the open state. If we start with (D), we know that the safe opens if we were in *diagonal-2*-state (and Christmas is saved) or the safe remains in its original state. If we do then action (A), the lock either opens or goes to *diagonal-2*-state if the original state was *adjacent-2* or stays in its original state. If the original state was *adjacent-2* and (A) brought it to *diagonal-2*, we must do (D) again to open the safe. So, after first taking action (D), then taking action (A), and then taking action (D), we are either done (the safe opened) or we are (still) in the state *1-light-on*. If we now press (1), we only know we will go out of the state *1-light-on*. Well, we already found out that the sequence $(D) \to (A) \to (D)$ leads to an open safe for all initial states other than *1-light-on*, so performing that sequence again will guarantee that the safe opens and Christmas is saved!

It remains to show that this is the only possible strategy to open the safe. With the help of Figure 2 we can easily disprove all of the other possible answers. For example, we can rule out $(1) \rightarrow (A) \rightarrow (D)$ in the following way: If the original state is *diagonal-2* then the lock would proceed to state *1-light-on* if we do action (1). Then applying (A) and (D) would leave the lock in state *1-light-on* and the safe remains locked. Similarly, we can rule out all of the other answers, except for answer 6.

Illustration: Friederike Hofmann

2 Let's Wrap It Up!

Author: Marieke Heidema (University of Groningen)

Challenge

In the days before Christmas, Santa's elves are working harder than ever. Wrapping paper is flying around in the wrapping room; tape is sticking to every surface imaginable. Unwrapped gifts are piled up on one side of the massive wrapping room, towering over all elves that are frantically cutting, taping, wrapping...

Between the rolls of lint and tape dispensers, there is an alarm sitting on one of the elves' desks. The alarm is counting down the hours they have left to wrap all the presents. Right now, only 10 hours are left... And the pile of presents? It contains 350 presents! Will the elves manage to wrap all presents in time?

There are 8 elves wrapping: Anna, Bea, Candice, Dan, and Eddy immediately start wrapping. Anna and Candice are experienced wrappers, each wrapping 10 presents per hour! Bea and Eddy are a bit slower and can only pack 7 presents per hour, and Dan wraps 3 per hour. After 2 hours of wrapping, Fince comes to help them out, but he can only wrap 6 presents per hour. When there are only 4 hours left on the alarm clock, Candice and Fince need to leave. With 2 hours on the clock, Geoffrey and Hugh come to help out. They can wrap 4 (Geoffrey) and 5 (Hugh) presents per hour, respectively. At the same time, Bea leaves and Fince comes back. Will there be any unwrapped presents left when the time on the alarm clock ends?

Possible Answers:

1. Yes, the elves ran out of time and there are exactly 50 unwrapped presents left!
2. Yes, the elves ran out of time and there are exactly 40 unwrapped presents left!
3. Yes, the elves ran out of time and there are exactly 30 unwrapped presents left!
4. Yes, the elves ran out of time and there are exactly 20 unwrapped presents left!
5. Yes, the elves ran out of time and there are exactly 10 unwrapped presents left!
6. No, the elves finished wrapping all the presents exactly in time!
7. No, the elves finished wrapping all the presents in time and could have even wrapped 10 extra presents, but no more!
8. No, the elves finished wrapping all the presents in time and could have even wrapped 20 extra presents, but no more!
9. No, the elves finished wrapping all the presents in time and could have even wrapped 30 extra presents, but no more!
10. No, the elves finished wrapping all the presents in time and could have even wrapped 40 extra presents, but no more!

Solution

The correct answer is: 8.

To compute the total amount of presents wrapped by each elf, we need the rate at which they wrap and the number of hours they spent wrapping. With that, we can use the following formula to compute the total number of presents wrapped by that elf:

$$\text{presents wrapped} \ = \ \text{wrapping rate} \ \times \ \text{hours spent wrapping.}$$

After the 10 hours are over, this is how many presents each elf could have wrapped:

Anna	$10 \times 10 = 100$
Bea	$7 \times 8 = 56$
Candice	$10 \times 6 = 60$
Dan	$3 \times 10 = 30$
Eddy	$7 \times 10 = 70$
Fince	$6 \times (4 + 2) = 36$
Geoffrey	$4 \times 2 = 8$
Hugh	$5 \times 2 = 10$

In total, this means that the elves could have wrapped 370 presents in those 10 hours. In other words, the 350 presents have all been wrapped and the elves could have even wrapped 20 more presents!

Illustration: Ivana Martić

3 The Elves' Circular Storages

Author: Lukas Protz
Project: MATH+

Challenge

Due to a recent increase in storms, the elves from two storage buildings want to connect their facilities with conveyor belts. This would allow them to transfer goods between the storages without having to go outside. Looking from above, the two storage buildings have the form of a circle, the first one with a radius of 50 meters and the second one with a radius of 70 meters. The centers of the two facilities are exactly 200 meters apart. The elves want to built two conveyor belts that are each, as presented in figure 3, tangent to the buildings and run exactly from one building to the other, but no further.

Unfortunately, there is a storm out there right now, such that the elves cannot measure the length of each of the conveyor belts. But they need to know the correct lengths of the two conveyor belts, to order enough material for the building process. Can the elves figure out the correct lengths without measuring, and if they can, what are the lengths of the conveyor belts 1 and 2 (rounded to full meters)?

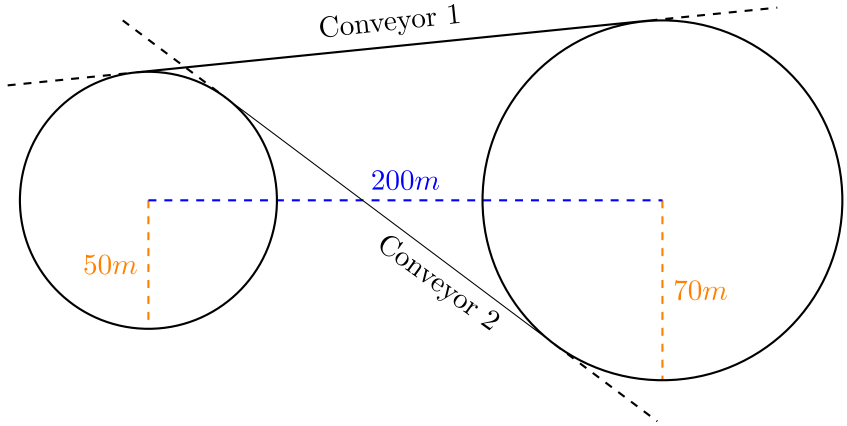


Figure 3: Top view of the buildings with the two conveyor belts drawn in.

Possible Answers:

1. It is not possible to determine the lengths.
2. Conveyor 1: 250 m, Conveyor 2: 233 m
3. Conveyor 1: 150 m, Conveyor 2: 220 m
4. Conveyor 1: 161 m, Conveyor 2: 233 m
5. Conveyor 1: 161 m, Conveyor 2: 160 m
6. Conveyor 1: 200 m, Conveyor 2: 200 m
7. Conveyor 1: 201 m, Conveyor 2: 233 m
8. Conveyor 1: 201 m, Conveyor 2: 160 m
9. Conveyor 1: 199 m, Conveyor 2: 233 m
10. Conveyor 1: 199 m, Conveyor 2: 160 m

Solution

The correct answer is: 10.

The general formula for the length l of the tangent corresponding to Conveyor 1 (see Figure 3) for radii r, r' and distance d between the center of the two circles is given by

$$l = \sqrt{d^2 - (r - r')^2}.$$

To prove this formula, we rotate Figure 3 around the center of the left circle such that Conveyor 1 is horizontal (see Figure 4).

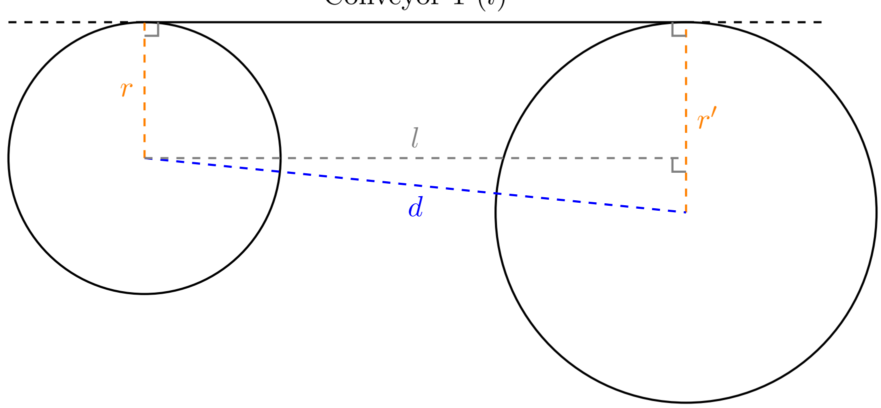


Figure 4: Rotated version of Figure 3 without Conveyor 2. The triangle formed by the gray, blue and part of the orange line is right angled.

Because Conveyor 1 is tangent to both circles, it is perpendicular to the drawn radii r and r'. By translating the line segment of length l of Conveyor 1 downwards by the amount of the smaller radius r one obtains a right angled triangle with side lengths l, $r - r'$ and hypotenuse d. This triangle really is right angled, as Conveyor 1 is perpendicular to the drawn radius r' and the translated line segment is parallel to Conveyor 1.

Thus by the Pythagorean theorem we get

$$l^2 + (r - r')^2 = d^2$$

or, rearranging

$$l = \sqrt{d^2 - (r - r')^2}$$

Inserting the given values, we arrive at

$$l = \sqrt{200^2 - (50 - 70)^2} = 60 \cdot \sqrt{11} \approx 199.$$

Similarly, by considering Figure 5, which again is a rotated version of Figure 3 around the center of the first circle, one can find another right angled triangle and conclude that for the length l of Conveyor 2 the formula

$$l = \sqrt{d^2 - (r + r')^2}$$

holds. Again, inserting the values we obtain

$$l = \sqrt{200^2 - (70+50)^2} = 160.$$

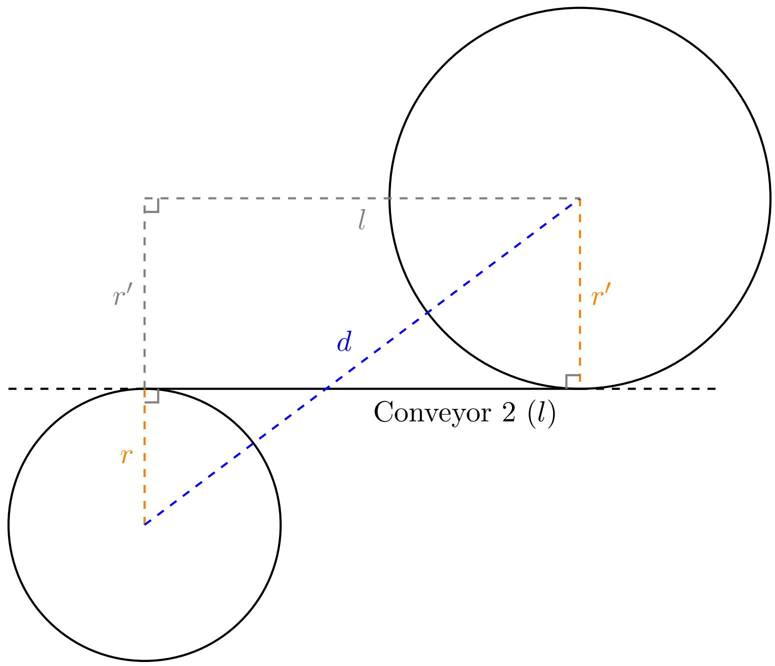


Figure 5: Rotated version of Figure 3 without Conveyor 1. The triangle formed by the gray, blue and gray-orange line is right angled.

The two derived formulas are so similar, that geometers introduced circles with signed radii (i.e. circles are allowed to have negative radii) to combine the two formulas into one:

We saw, that there are essentially two ways of drawing a tangent between two circles (up to symmetry). By introducing signed radii, one can refer to the case of Conveyor 1, if the two radii have the same sign, and to Conveyor 2, if the radii have opposite signs. In this way, the length of the Conveyors can always be calculated as:

$$l = \sqrt{d^2 - (r - r')^2}.$$

Indeed, if we want to calculate the length of Conveyor 1, we can choose both radii to be positive or both to be negative and get:

$$l = \sqrt{200^2 - (50-70)^2} = \sqrt{200^2 - (-50-(-70)^2)} \approx 199.$$

For Conveyor 2 we can either choose r to be positive and r' to be negative or the other way round:

$$l = \sqrt{200^2 - (50-(-70))^2} = \sqrt{200^2 - (-50-70)^2} = 160.$$

There even is a whole branch of geometry called Laguerre geometry, that is devoted to the study of transformations, that map circles with signed radii to circles with signed radii, such that the tangential distance (length of the Conveyors) stays the same.

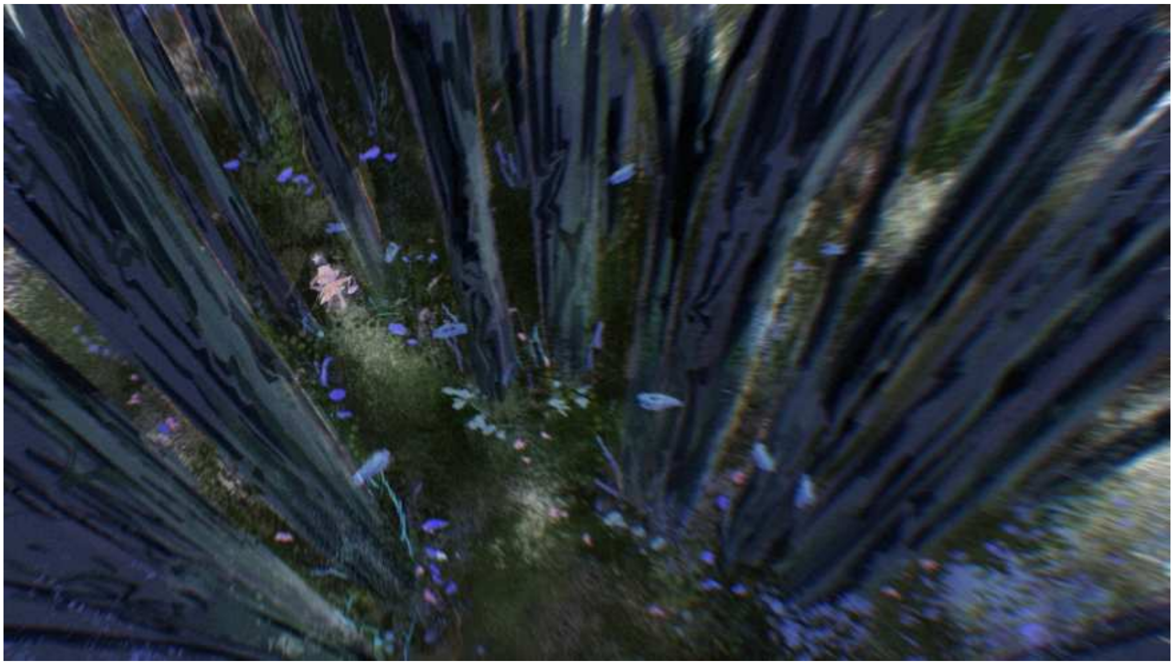


Illustration: Zyanya Santuario

4 Lost in the Magical Forest

Author: Marvin Lücke
Project: EF1-10

Challenge

The elf Gilfi sets out to gather mushrooms in the magical forest for a Christmas feast. This magical forest not only has the best mushrooms far and wide but also a very special structure: at the forest's entrance, there are three paths. Each of the three paths leads to a different clearing. At each clearing, there are two new paths, again each leading to a new different clearing. If one continues onward, this pattern repeats: every clearing you reach opens up two new paths to other clearings. Gilfi excitedly runs into the magical forest, choosing a path at each clearing, with probability $1/3$ for each path - either one of the two new paths or the path he arrived on. After gathering some mushrooms, he realizes he's lost in the magical forest! But in fact, he ends up in one of the three clearings that can be reached directly from the entrance via one of the three paths. What is the probability that he will eventually find his way back to the entrance from this point if he always chooses one of the three paths at a clearing at random?

Hint: One way to start is to consider the probability for any given clearing of moving towards the forest entrance or deeper into the forest.

Possible Answers:

1. 0
2. 1/10
3. 1/6
4. 1/4
5. 1/3
6. 1/2
7. 2/3
8. 3/4
9. 4/5
10. 1

Project Reference

Many physical, biochemical, and social processes are described by stochastic particle-based models. This task can be interpreted as the random movement of a particle on a network.

Solution

The correct answer is: 6.

Let P be the probability that Gilfi will eventually find his way back to the entrance from his current position of being 1 path away from the entrance.
The probabilty P can also be interpreted in another way: At each clearing Gilfi is at on his journey through the forest, P is the probability of Gilfi eventually ending up one path closer to the entrance compared to his current position. This implies, that the probability of Gilfi ending up n paths closer to the entrance, for some natural number n, is P^n. This means, that we can set up the following equation for P:

$$P = \frac{1}{3} + \frac{2}{3}P^2.$$

Indeed, Gilfi eventually reaches the entrance if he either chooses the correct path directly, which has probability $1/3$ or if he goes deeper into the forest, which has probability $2/3$ and then manages to eventually end up 2 paths closer to the entrance, which has probability P^2. The above equation is quadratic and can be solved using the *abc formula* or *p-q formula*. In standard form, the equation is:

$$\frac{2}{3}P^2 - P + \frac{1}{3} = 0.$$

The solutions of the quadratic equation are:

$$P = 1 \quad \text{and} \quad P = 0.5.$$

Intuitively, it is clear that $P = 1$ cannot be true. Proving this, however, turns out to be more complicated.
Let $\tilde{Q}(k)$ be the probability of following event: Gilfi returns to the entrance of the forest by using exactly k paths. Since these events are clearly independent, the probability P of returning to the entrance at all is given by the sum of the $\tilde{Q}(k)$:

$$P = \tilde{Q}(1) + \tilde{Q}(2) + \dots.$$

However, to return to the entrance, Gilfi needs to use an odd number of paths. Therefore,

$$\tilde{Q}(2k) = 0.$$

By setting $Q(k) \coloneqq \tilde{Q}(2k+1)$ we thus want to show:

$$P = Q(0) + Q(1) + Q(2) + \dots < 1.$$

To reach the entrance using $2k+1$ paths, Gilfi needs to move closer to the entrance once more than going further into the forest and he must not reach the entrance before using the $2k+1$-st path. Each such sequence of paths has probability $(1/3)^{k+1}(2/3)^k$. The number of these sequences of paths is given by the k-th Catalan number

$$C_k \coloneqq \frac{(2k)!}{(k+1)!k!}.$$

We will not prove this fact, as the solution is already quite long, but we encourage interested readers to search "Catalan numbers" online for a proof.

Given this, we can examine the probabilities $Q(k)$ closer:
For $k = 0$ we have

$$Q(0) = C_0 \frac{1}{3} = \frac{1}{3}.$$

For $k > 0$ we can do the following:

$$C_k \left(\frac{1}{3}\right)^{k+1} \left(\frac{2}{3}\right)^k = \frac{1}{3} C_k \left(\frac{2}{9}\right)^k = \frac{1}{3} C_k \frac{1}{4^k} 4^k \left(\frac{2}{9}\right)^k = \frac{1}{3}\frac{1}{4} \cdot C_k \frac{1}{4^{k-1}} \left(\frac{8}{9}\right)^k.$$

Turning our attention to the term $C_k(1/4^{k-1})$ we can deduce that this quantity is always less or equal to 1. For $k > 1$ it even holds $C_k(1/4^{k-1}) < 1$:

Claim: $(2k)! < (k+1)!k!4^{k-1}$ for $k \geq 2$ and $(2k)! = (k+1)!k!4^{k-1}$ for $k = 1$.

Proof. At first, let us confirm the claim for $k = 1$:

$$(2 \cdot 1)! = 2 = 2 \cdot 1 \cdot 1 = (1+1)!1!4^{1-1}.$$

Now we will prove the claim for $k \geq 2$ via a method called induction:
For $k = 2$ we have

$$(2 \cdot 2)! = 24 < 48 = 6 \cdot 2 \cdot 4 = (2+1)!2!4^{2-1}.$$

Thus, the claim holds for $k = 2$.
Now, assume that we already know that the claim holds for some number $k \geq 2$. We will explain later why this assumption is valid. We now want to show, that it also holds for $k+1$:

$$\begin{aligned}(2(k+1))! &= (2k+2)(2k+1)(2k)! \\ &< (2k+2)(2k+1)(k+1)!k!4^{k-1} \\ &< 4(k+2)(k+1)(k+1)!k!4^{k-1} \\ &= (k+1+1)!(k+1)!4^{k+1-1}\end{aligned}$$

Hence, if the claim holds for some k it also holds for $k+1$.
Finally, starting with $k = 2$ we can use the above argument to deduce that the claim is true for $k = 3$. Then again, we can use this argument to show that the claim is true for $k = 4$. This process can be continued. Thus, for an arbitrary k we can deduce that the claim holds for k be repeatedly applying the above argument until we reach the number k. This concludes the proof. □

This claim can be used to obtain the following inequality:

$$Q(0) + Q(1) + Q(2) + Q(3) + \ldots < \frac{1}{3} + \frac{1}{12}\left(\frac{8}{9}\right)^1 + \frac{1}{12}\left(\frac{8}{9}\right)^2 + \frac{1}{12}\left(\frac{8}{9}\right)^3 + \ldots$$

Factoring out the term $1/12 \cdot 8/9$ from the second summand onward on the left hand side, we obtain a geometric series:

$$\frac{1}{12}\frac{8}{9}\left(\left(\frac{8}{9}\right)^0 + \left(\frac{8}{9}\right)^1 + \left(\frac{8}{9}\right)^2 + \ldots\right) = \frac{1}{12}\frac{8}{9}\left(\frac{1}{1-\frac{8}{9}}\right) = \frac{2}{3}.$$

This allows us to conclude

$$P < \frac{1}{3} + \frac{2}{3} = 1,$$

excluding the possibility of $P = 1$ and confirming that $P = 0.5$.

Additional Information: This is the return probability of a random walk on a so-called Bethe lattice, see Fig. 6. The proof above works similarly in the general case where each node (clearing in the forest) has degree z. (In this task, $z = 3$ since there are three paths leading to each clearing.) The return probability is then $P(1) = \frac{1}{z-1}$.

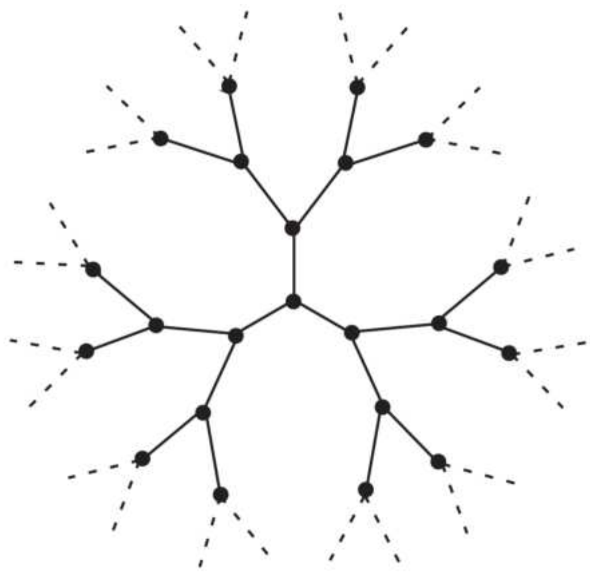


Figure 6: Bethe lattice. (Source: ben-Avraham et al., J. Phys.: Condens. Matter, 2007)

Illustration: Ivana Martić

5 Coating cookies with glaze

Author(s): Rudolf Straube (Heinrich-Hertz-Gymnasium Berlin) and
Arthur Straube (ZIB)
Project: AA1-18

Challenge

The elves Bob and Sieglinde were sitting happily playing chess on Advent Day. Suddenly the clock struck, heralding their shift at the Christmas factory. Their task in the Christmas factory is to coat cookies, consisting of 10 square pieces, of the rectangular shapes 1×10 and 2×5 with glaze (see figure 7).

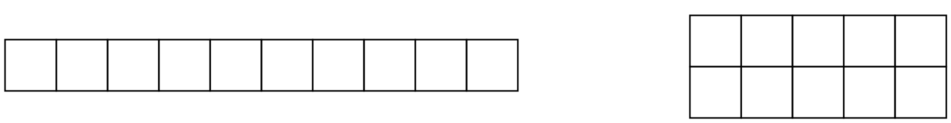


Figure 7: The cookies of the shapes 1×10 (left) and 2×5 (right).

To ensure that the cookies remain unique in their decoration, Bob and Sieglinde coat pieces of the cookies with glaze according to the "Christmas chess principle". Like chess kings hitting the up to eight squares around them on a chess board, they only apply the glaze to the individual squares in such a way that no two squares covered with glaze touch each other on an edge or corner. At the same time, however, there must be no space left for another glazed square on the entire cookie. Optimization is not an issue here. It is therefore quite possible that in one arrangement more and in another fewer glaze squares would have fitted on the entire rectangular cookie.

How many different ways are there to glaze the cookies in the shapes 1×10 and 2×5? The answer must be given in pairs of numbers (M, N), where M stands for the shape 1×10 and N for the shape 2×5.

Note: Consider only the final configurations as different. Moreover, configurations that can be transformed into each other by rotating the cookie are also considered as different.

Possible Answers:

1. $(9, 12)$
2. $(12, 12)$
3. $(12, 16)$
4. $(12, 20)$
5. $(12, 24)$
6. $(12, 36)$
7. $(16, 16)$
8. $(16, 20)$
9. $(16, 24)$
10. $(16, 36)$

Project Reference:

This MATH$^+$ project focuses on biochemical oscillators. In particular, the characteristic intracellular conditions imply that chemical reactions take place in very small volumes. At these microscopic scales, the discrete nature of the reacting components becomes crucial. This means that the biochemical processes taking place are inherently stochastic and classical macroscopic models of reaction kinetics are often not applicable. The proposed challenge of counting configuration possibilities is a simple example that borders directly on the field of stochastics and probability theory. The challenge shows that even in relatively simple situations, direct counting of possibilities can become difficult. However, by skillfully using mathematical methods, the effort can be considerably reduced and more general solutions can be obtained.

Solution

The correct answer is: 8. $(\mathbf{16}, \mathbf{20})$

1. Direct counting.

The obvious way is to look at all the different configurations directly and count them. All possible configurations are shown in the figures 8 and 9.

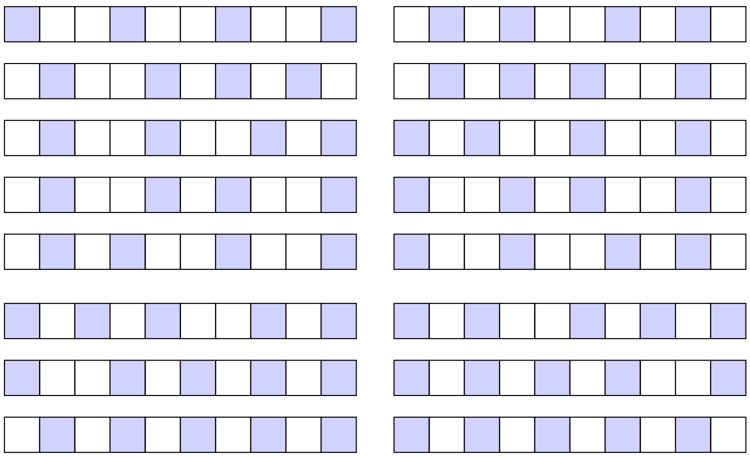


Figure 8: All 16 configurations for cookies of the shape 1×10.

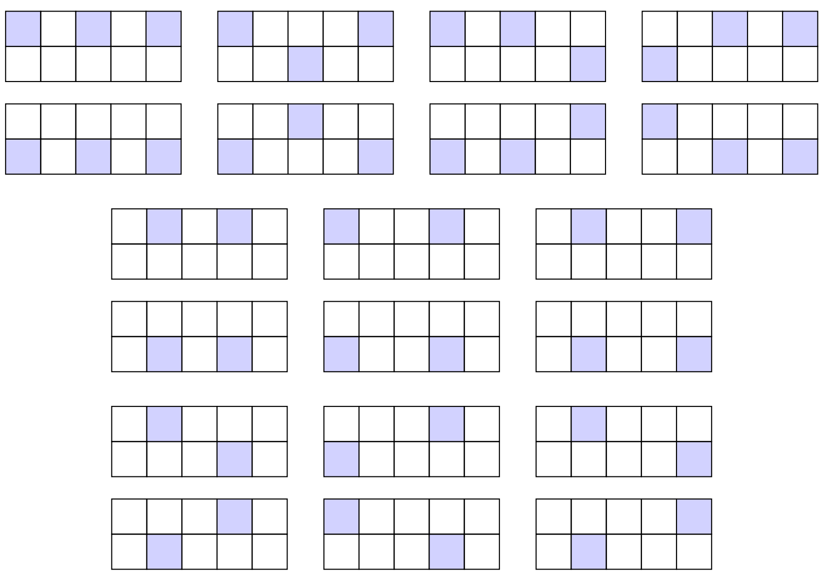


Figure 9: All 20 configurations for cookies of the shape 2×5.

This method works perfectly for relatively small cookies. For larger cookies, however, it quickly becomes complex. This is where the power of mathematics can provide a much more elegant and efficient solution.

2. Recursion formula.

Consider a slightly more general configuration $2 \times n$. Let N_n be the number of independent ways to cover cookies of shape $2 \times n$ with glaze. Let us start from the left and coat the cookie step by step from left to right according to the given rules. At the beginning we have 4 possibilities to cover one of the four green marked squares (see figure 10) with glaze. Considering the four possible options, the task can be reduced to a cookie that is similar to the initial one, but slightly shorter, with a shape of $2 \times (n-2)$ (bottom left) or a shape of $2 \times (n-3)$ (bottom right). In figure 10, the square boxes that are covered with glaze are marked in blue and the boxes that are blocked by this are marked in red. In general, the configuration N_n can be reduced to either two options in which a square box in the first column is coated with glaze (figure 10, bottom left), or to two options in which a square box in the second column is coated (figure 10, bottom right). Accordingly, this results in $2N_{n-2}$ for the number of possibilities in the first case and to $2N_{n-3}$ in the second case.

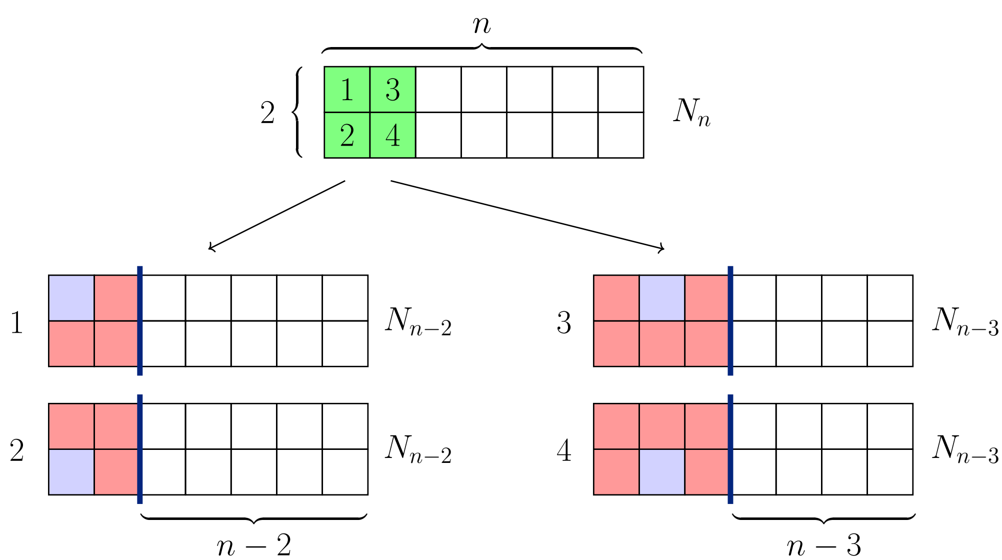


Figure 10: There exist 4 possibilities to reduce a cookie of shape $2 \times n$ (top) to smaller cookies of the shape $2 \times (n-2)$ (bottom left) or $2 \times (n-3)$ (bottom right).

Summarizing the results, a recursion formula of the following form is obtained:

$$N_n = 2(N_{n-2} + N_{n-3}) \quad (n \geq 4)\,. \tag{1}$$

To use it, one also needs the starting values N_1, N_2 and N_3. These can be found quite easily by directly counting the number of possible configurations (see figure 11), which leads in our cases to the values $N_1 = 2$, $N_2 = 4$ and $N_3 = 6$.
Using these initial values and the formula (1), one successively finds $N_4 = 12$, $N_5 = 20$, $N_6 = 36$ etc. In the given task, we are interested in $N_5 = 20$, which agrees with the results of direct counting.

For the number M_n of ways to coat the cookies of the shape 1×10 with glaze, the derivation is similar. Here the recursion formula is given by

$$M_n = M_{n-2} + M_{n-3} \quad (n \geq 4) \tag{2}$$

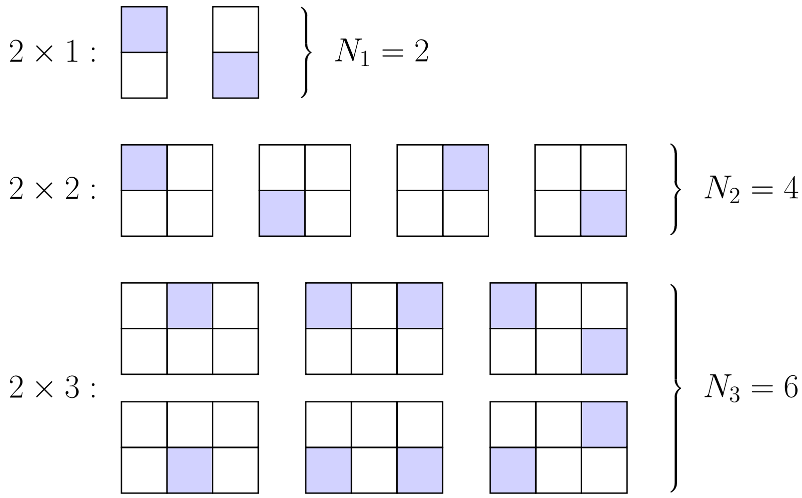


Figure 11: Illustration of the configurations for the initial conditions: $N_1 = 2$, $N_2 = 4$, $N_3 = 6$.

with the initial values $M_1 = 1$, $M_2 = 2$, $M_3 = 2$. This yields: $M_4 = 3$, $M_5 = 4$, $\ldots$, $M_9 = 12$, $M_{10} = 16$, $M_{11} = 21$, and so on. The number required in the problem statement is $M_{10} = 16$.

Illustration: Zyanya Santuario

6 Workbench Hubbub

Author Tobias Paul (HU Berlin)
Project: AA3-18

Challenge

In order to wrap the Christmas presents appropriately, Father Christmas has provided wrapping paper in the colours red and blue this year.
A total of 500 Christmas elves sit at a workbench with their presents to be wrapped in pairs facing each other (i.e. 250 on each side). The elves proceed as follows:

- Each elf wraps their present at the same time as the elf sitting opposite.
- The pair of elves at the start of the workbench wrap their presents first. Each elf decides for themselves whether to use red or blue wrapping paper.
- Then the next pair of elves sitting opposite each other wraps their presents, followed by the third pair and so on. Wrapping in pairs therefore starts at one end of the workbench and gradually continues to the other end.

On day 1, each elf sees the packaging of the previous elf on the opposite side out of the corner of their eye and decides to use exactly the same colour for the wrapping paper.
The first pair decides on one of the two colours with a probability of 1/2, completely independently of each other.

On day 2, each elf sees the packaging of the previous 2 elves on the opposite side out of the corner of their eye. If the two packages are the same colour, they will choose the same colour. However, if he sees different colours, he will choose one of the two colours completely independently with a probability of 1/2. The first 2 pairs at the beginning of the table decide completely independently of each other with a probability of 1/2 for the colour of their wrapping paper.

Shortly before Christmas, Santa is now faced with the following questions: What is the probability that both elves of the last pair of elves (elves number 250) at the table will use the same colour for their wrapping paper (a) on day 1 and (b) on day 2?
Watch out, Father Christmas has rounded the possible answers to the 2nd decimal place.
Note: For day 2, for example, it could be helpful to think about the possibilities for the first five pairs.

Possible Answers:

1. (a): 1 (b): 1
2. (a): 1 (b): 0.5
3. (a): 1 (b): 0
4. (a): 0.5 (b): 1
5. (a): 0.5 (b): 0.5
6. (a): 0.5 (b): 0.33
7. (a): 0.5 (b): 0
8. (a): 0 (b): 1
9. (a): 0 (b): 0.5
10. (a): 0 (b): 0.33

Project Reference:

In game theory, there is a so-called "adaptive play" process. In the specific case of the coordination game with two players and two options (in which both players want to agree, i.e. choose the same option), this process does the following: To make the decision for the next game round, each player is allowed to look at a number $s \geq 1$ of the opponent's moves from the last h game rounds. In the task, this corresponds to $s = 1$ and $h \in \{1, 2\}$. The best response to what is seen in this game is then to copy the move (because the players want to match).
In the project, we want to investigate variants of this process using methods from population genetics and thus make possible statements about e.g. frequencies of moves over time.

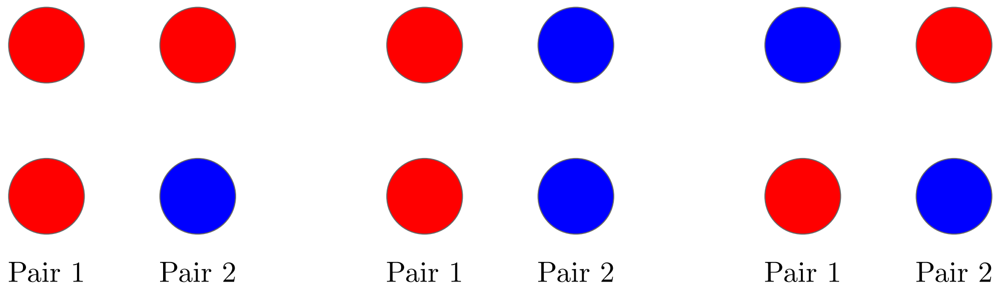


Figure 12: The three possibilities (along with swapped roles for red and blue) for Case 1.

Solution

The correct answer is: 4.

We solve the problem using simple combinatorics. On Day 1, the first pair of elves uses the same wrapping paper with a probability of $1/2$, because both elves use red wrapping paper with a probability of $(1/2) \cdot (1/2) = 1/4$ and blue wrapping paper with the same probability, $(1/2) \cdot (1/2) = 1/4$.
The second pair chooses their wrapping paper color just like the first pair, so the probability that they use the same paper is also $1/2$. This argument can be applied iteratively, and we see that the last pair of elves also uses the same wrapping paper with a probability of $1/2$.

On Day 2, things are a bit trickier. First, we observe that as soon as two adjacent pairs of elves use the same wrapping paper, all subsequent elves will also use the same wrapping paper. In mathematics, this is referred to as an *equilibrium*. Instead of analyzing the probability of the same wrapping paper, we examine the complementary probability. The four elves in the first two pairs choose their wrapping paper colors independently of each other, resulting in $2^4 = 16$ possible combinations. Of these, exactly 2 combinations have all four elves using the same wrapping paper. Thus, there is a probability of $14/16 = 7/8$ that these four elves do not match. Next, we consider how the wrapping paper of the third pair could look, given that the colors of the first and second pair are not the same.
To do this, we divide the wrapping choices of the first two pairs into two cases:

Case 1: There is a color that appears on both sides of the workbench (see Figure 12). This occurs in the following situations: 1. when the wrapping paper of one pair matches in color, but the other pair does not, 2. when the wrapping paper of both pairs matches, but the first pair uses a different color than the second pair, and 3. when the wrapping paper of neither pair matches, but the colors on each side do not repeat. In the case of 1, there is a probability of exactly $1/2$ that the third pair uses matching wrapping paper, and the fourth pair has a probability of at least $1/4$ of adopting the colors of the third pair (specifically, in cases where both elves mimic the color of the third pair). Thus, equilibrium is reached at pairs 3 and 4 with a probability of at least $1/8$.
In the case of 2 or 3, it can be similarly shown that equilibrium at pairs 3 and 4 is reached with a probability of at least $1/8$. In each case, the fourth pair has at least a $1/8$ probability of having the same wrapping paper as the third pair. This means the probability of differing colors among the third and fourth pairs is at most $7/8$. In

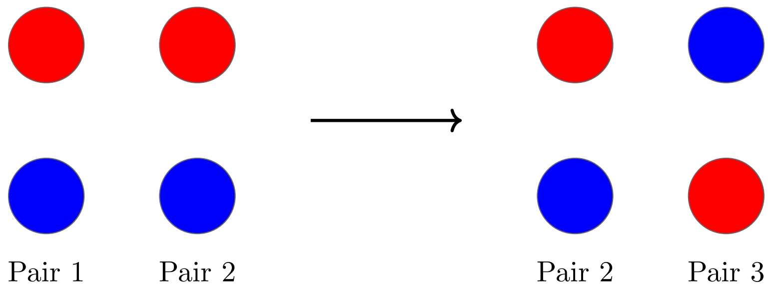


Figure 13: The initial configuration in Case 2 and the configuration for pairs 2 and 3 (from which pair 4 can choose their wrapping).

particular, the fourth and fifth pairs also have a probability of at most 7/8 of having differing colors. This is because, with at least 1/8 probability, pairs 3 and 4 all use the same color, and hence pair 5 does as well.

Case 2: The colors on both sides of the workbench are different, but the colors within each side match for the first and second elves. In this case, the wrapping colors of the second and third pairs will form a configuration from Case 1 (see Figure 13).

In both cases, it follows that pairs 4 and 5 have at most a probability of $(7/8) \cdot (7/8)$ of using different wrapping paper. Considering pairs 4 and 5 as new starting pairs, we can apply this argument to pairs 7 and 8, showing that the probability of differing wrapping paper for pairs 7 and 8 is at most $(7/8)^3$. In general, for pairs at positions $3k+1,\ 3k+2$ (where k is a natural number), the maximum probability of differing wrapping paper is $(7/8)^{k+1}$.
Pairs at positions $247 = 3 \cdot 82 + 1$ and $248 = 3 \cdot 82 + 2$ thus have a probability of at most $(7/8)^{83} \approx 0$ of differing colors. Similarly, pair 250 has at most a probability of $(7/8)^{83}$ of differing wrapping paper. Therefore, the final pair uses the same wrapping paper with a rounded probability of 1.

Illustration: Friederike Hofmann

7 Santa Cargo

Author: Lukas Neubauer

Challenge

The company Santa Cargo is responsible for shipping all the presents. Of course, Santa Cargo wants to minimize the effort in order to deliver all gifts on time and in the optimal way. Can you help the company?

For the next shipment Santa Cargo has to send $N = 10$ gifts. Traditionally, they use sleds to transport the cargo. A sled always comes with an initial effort of one elf-watt (the standard unit of measuring effort at the North Pole). Moreover, there is an additional effort depending on the number of gifts.

A sled can carry up to 10 gifts, but it is best balanced to carry three gifts. Depending on the number of gifts n, the effort is given by $(n-3)^2$ elf-watt. This means that a sled carrying only one gift takes more effort than a sled loaded with three gifts. That is because Santa Cargo wants to prevent underloaded or even empty sleds. So the effort by a single sled carrying n gifts can be calculated by:

$$e(n) := 1 + (n - 3)^2.$$

How many sleds should Santa Cargo load and how should the presents be divided up in order to ship the ten gifts with minimum effort?

A possible answer should have the form $(n_1, ..., n_k)$ where k is the number of sleds and n_i is the number of gifts on sled i. So, for example, the answer (10) has one sled carrying all the gifts, but $(2, 5, 3)$ has three sleds, the first one loaded with two gifts, the second has five gifts and the third carries three gifts. Also, note that a gift cannot be cut into several pieces.

Possible Answers:

1. (10)
2. $(10, 0)$
3. $(5, 4, 3)$
4. $(2, 3, 2, 3)$
5. $(2, 2, 2, 2, 2)$
6. $(7, 3)$
7. $(4, 4, 2)$
8. $(1, 2, 3, 4)$
9. $(5, 5)$
10. None of the above. The right answer is ...

Solution

The correct answer is: 10.

Since sleds carrying exactly three gifts require the least effort, we aim to use three sleds: two with three gifts each and one with four. This configuration gives a total effort of:

$$2e(3) + e(4) = 2 + 2 = 4.$$

We now demonstrate that this is the optimal solution by calculating the effort for different configurations with k of the sleds carrying three gifts. First, we observe that any sled carrying fewer or more than three gifts incurs an effort of at least 2 elf-watts.

Case 1: $k = 0$: Here we use no sled with three gifts. Therefore, to not exceed the 4 elf-watts limit, at most two sleds can be used, as each sled has an effort of 2 elf-watts or higher. Consequently, at least one sled would need to carry five or more gifts, resulting in a minimum effort of:

$$e(5) = 1 + (5 - 3)^2 = 5.$$

Thus, using no sleds with exactly three gifts leads to a total effort exceeding 4 elf-watts. Therefore, at least one sled must carry exactly three gifts in an optimal solution.

Case 2: $k = 1$: If one sled carries three gifts, which has an effort of one, then only one additional sled can be used without exceeding the 4 elf-watt limit. This second sled would need to carry seven gifts to cover all ten gifts:

$$e(7) = 1 + (7 - 3)^2 = 17.$$

This by far exceeds the limit of 4 elf-watts. Therefore, at least two sleds must carry three gifts in an optimal solution.

Case 3: $k \geq 2$: For the total effort to remain at 4 elf-watts or less, the only viable configurations are:

- One additional sled carrying three gifts (for a total of nine gifts),
- two additional sleds each carrying three gifts (for a total twelve gifts), or
- one additional sled carrying four gifts (for a total of ten gifts).

The first two options result in either nine or twelve gifts, which is inconsistent with the requirement to transport exactly ten gifts. The only feasible configuration is $(3, 3, 4)$, which therefore is optimal.

Alternative and General Solution (Using Derivatives)

A single sled carrying n gifts incurs the effort

$$e(n) := 1 + (n - 3)^2.$$

Here, the 1 represents the initial effort, and $(n-3)^2$ accounts for the additional effort, which depends on the number of gifts n. The sum of all efforts gives the total effort:

$$E(n_1, ..., n_k) := \sum_{i=1}^{k} e(n_i) = \sum_{i=1}^{k} \left[1 + (n_i - 3)^2\right].$$

Now, we aim to minimize the effort while transporting a total of $N = 10$ gifts:

$$\min_{\sum_{i=1}^{k} n_i = N} E(n_1, ..., n_k).$$

This is an example of competing energies/efforts. To minimize the effort, the number of sleds must be minimized. However, this increases the effort for the individual sled.

On the other hand, one could minimize the effort for a single sled (by always loading three gifts onto a sled), but this increases the number of sleds.

To find the minimal effort, these two competing mechanisms must be carefully balanced.

For the given solutions, we can manually compute the respective effort:

1. $E(10) = 1 + (10-3)^2 = 50$
2. $E(10, 0) = 1 + (10-3)^2 + 1 + (0-3)^2 = 60$
3. $E(5, 4, 3) = 1 + (5-3)^2 + 1 + (4-3)^2 + 1 + (3-3)^2 = 8$
4. $E(2, 3, 4, 3) = 1 + (2-3)^2 + 1 + (3-3)^2 + 1 + (4-3)^2 + 1 + (3-3)^2 = 6$
5. $E(2, 2, 2, 2, 2) = 5 \cdot (1 + (3-2)^2) = 10$
6. $E(7, 3) = 18$
7. $E(4, 4, 2) = 6$
8. $E(1, 2, 3, 4) = 10$
9. $E(5, 5) = 10$

So far, solution 7 has the lowest effort among the given possibilities. But is there a better solution?

First, we note that if we want to use only one sled, there is not much choice, as there is only one way to load the sled: placing all ten gifts on it, which results in an effort of

$$E(10) = 1 + (10-3)^2 = 50.$$

But what if we want to use two sleds? If we load the sleds with (n_1, n_2), we obtain

$$E(n_1, n_2) = 1 + (n_1 - 3)^2 + 1 + (n_2 - 3)^2.$$

If we additionally use the constraint $n_1 + n_2 = N$, where N denotes the total number of gifts to be transported (in our specific case $N = 10$), we get

$$E(n_1) = 1 + (n_1 - 3)^2 + 1 + (N - n_1 - 3)^2.$$

If this is to be minimized, the derivative with respect to n_1 must be zero. Therefore, we compute the derivative with respect to n_1:

$$E'(n_1) = 2(n_1 - 3) - 2(N - n_1 - 3) \stackrel{!}{=} 0$$

Rearranging gives

$$n_1 = \frac{N}{2}$$

and thus, by the constraint, also $n_2 = \frac{N}{2}$. This means that we should distribute the gifts evenly between the sleds if we use only two sleds. Compare, for example, the computed efforts:

2. $E(10, 0) = 60$
6. $E(7, 3) = 18$
10. $E(5, 5) = 10$

Even though one sled in $E(7, 3)$ is perfectly loaded, it causes much more effort than evenly distributing the load across both sleds in $E(5, 5)$.

The assumption suggests that even when using a general number of k sleds, an even distribution is optimal. We will prove this now:
Let $n_1, \ldots, n_k$ be the number of gifts on the k sleds. Then, we have

$$E(n_1, \ldots, n_k) = \sum_{i=1}^{k} \left[1 + (n_i - 3)^2\right].$$

Using the constraint $\sum_{i=1}^{k} n_i = N$, we can eliminate the variable n_k from the equation for E, yielding

$$E(n_1, \ldots, n_{n_{k-1}}) = \sum_{i=1}^{k-1} \left[1 + (n_i - 3)^2\right] + 1 + \left(N - \sum_{i=1}^{k-1} n_i - 3\right)^2.$$

To minimize E, all partial derivatives in the direction of n_i must be zero:

$$\partial_{n_i} E(n_1, \ldots, n_{k-1}) \stackrel{!}{=} 0$$

for $i \in \{1, \ldots, k-1\}$. Here, ∂_{n_i} represents the derivative with respect to the variable n_i. This gives us the equations:

$$\partial_{n_i} E(n_1, \ldots, n_{k-1}) = 2(n_i - 3) - 2\left(N - \sum_{i=1}^{k-1} n_i - 3\right) \stackrel{!}{=} 0.$$

Now, we consider two of the above equations for two arbitrary indices l and m:

$$2(n_l - 3) - 2\left(N - \sum_{i=1}^{k-1} n_i - 3\right) \overset{!}{=} 0,$$
$$2(n_m - 3) - 2\left(N - \sum_{i=1}^{k-1} n_i - 3\right) \overset{!}{=} 0.$$

Subtracting these two equations leads to

$$2(n_l - 3) - 2(n_m - 3) = 0,$$

or equivalently,

$$n_l = n_m.$$

Since l and m were arbitrary, we conclude:

All sleds should be loaded evenly.

It remains to show that the values $n_1 = \frac{N}{k}, \ldots, n_k = \frac{N}{k}$ indeed yield a minimum of E. Consider numbers $\delta_1, \ldots, \delta_k$ that represent deviations in the number of gifts $n_1, \ldots, n_k$ from $\frac{N}{k}$. Using the constraint, we have $\delta_k = \frac{N}{k} - \sum_{i=1}^{k-1} \delta_i$. Applying the binomial expansion, we obtain

$$\begin{aligned}
E\left(\frac{N}{k} + \delta_1, \ldots, \frac{N}{k} + \delta_{k-1}, \frac{N}{k} - \sum_{i=1}^{k-1} \delta_i\right) &= k + \sum_{i=1}^{k-1} \left(\frac{N}{k} + \delta_i - 3\right)^2 + \left(\frac{N}{k} - \sum_{i=1}^{k-1} \delta_i - 3\right)^2 \\
&= k + \sum_{i=1}^{k-1} \left(\frac{N}{k} - 3\right)^2 + 2\left(\frac{N}{k} - 3\right) \sum_{i=1}^{k-1} \delta_i + \sum_{i=1}^{k-1} \delta_i^2 \\
&\quad + \left(\frac{N}{k} - 3\right)^2 - 2\left(\frac{N}{k} - 3\right) \sum_{i=1}^{k-1} \delta_i + \left(\sum_{i=1}^{k-1} \delta_i\right)^2 \\
&= E\left(\frac{N}{k}, \ldots, \frac{N}{k}\right) + \underbrace{\sum_{i=1}^{k-1} \delta_i^2}_{\geq 0} + \underbrace{\left(\sum_{i=1}^{k-1} \delta_i\right)^2}_{\geq 0} \\
&\geq E\left(\frac{N}{k}, \ldots, \frac{N}{k}\right)
\end{aligned}$$

Any deviation from the found solution leads to a higher total effort, confirming that the solution is indeed a minimum.

Now, the remaining question is: How many sleds should we use? We know that we should evenly distribute the gifts among the k sleds, each carrying $10/k$ gifts. Given that we must distribute 10 gifts among k sleds, we obtain a function that depends only on k:

$$\begin{aligned}
\tilde{E}(k) \coloneqq E(10/k, \ldots, 10/k) &= \sum_{i=1}^{k} \left[1 + \left(\frac{10}{k} - 3\right)^2\right] \\
&= k\left[1 + \left(\frac{10}{k} - 3\right)^2\right] \\
&= \frac{100}{k} + 10k - 60.
\end{aligned}$$

To minimize this function, we set its derivative with respect to k to zero:

$$\tilde{E}'(k) = -\frac{100}{k^2} + 10 \stackrel{!}{=} 0$$

which simplifies to $k = \sqrt{\frac{100}{10}} = \sqrt{10} \approx 3{,}162\ldots$. We omit the proof that this is indeed a minimum, but it can be easily verified using the second derivative test.

Thus, we should use three sleds, each carrying three gifts, with one sled carrying an additional gift to account for all ten gifts. Santa Cargo should load the sleds as follows: **(3,3,4)** to minimize effort. The order of the sleds does not matter.

Illustration: Zyanya Santuario

8 Wild Reindeer

Author: Margarita Kostré (ZIB)
Project: EF5-2

Challenge

Santa Claus has acquired new wild reindeer that he would like to use. However, he lacks the time to train them and has only limited knowledge about their movement. He knows that the position x_t of the reindeer at time step t and the velocity v_t at time step t follow some rules. To describe these rules properly, he introduces a coordinate system on the ground (plane). Then he can write down iterative equations for the velocity v_t and position x_t:

$$v_{t+1} = -0.5v_t - ax_t$$

$$x_{t+1} = x_t + v_{t+1},$$

where the so called fear value a is a positive real number that depends on the reindeer.The fear value describes whether a reindeer is tame and will eventually stop running away from Santa. If a reindeer's fear value a is favorable, the reindeer will become arbitrarily slow and will remain slow over time. Santa can then catch it and bring it home. However, there are some reindeer that cannot overcome their fear. These reindeer keep running faster and faster and cannot be caught.

We are looking for all fear values a that a reindeer can have so that Santa can catch it. It is assumed that the reindeer start with an initial velocity $v_0 = (0, 0)$ at the position $x_0 = (1, 1)$.

Hints and remarks:

Note, that the multiplication of a number λ and a point (x_1, x_2) is defined by $\lambda(x_1, x_2) = (\lambda x_1, \lambda x_2)$. The addition or subtraction is defined coordinate wise: $(x_1, x_2) \pm (y_1, y_2) = (x_1 \pm y_1, x_2 \pm y_2)$.

To solve this challenge it can be helpful to make use of the absolute value. The absolute value represents the distance of a number to 0 and is defined by

$$|x| = \begin{cases} x & \text{, wenn } x \geq 0 \\ -x & \text{, wenn } x < 0 \end{cases} .$$

For the absolute value and two real numbers a, b, the following rules hold:

$$|a + b| \leq |a| + |b|$$

$$|a - b| \geq ||a| - |b||$$

$$|ab| \leq |a||b|$$

The term “become and remain arbitrarily slow” in this challenge is meant as follows: For any positive number $\epsilon > 0$ there is a time t_0 such that if $t \geq t_0$, then each component v of the velocity v_t fulfills $|v| < \epsilon$.

Possible Answers

1. $0 < a < 0.1$
2. $0 < a < 0.2$
3. $0 < a < 0.3$
4. $0 < a < 0.4$
5. $0 < a < 0.5$
6. $0 < a < 0.6$
7. $0 < a < 0.7$
8. $0 < a < 0.8$
9. $0 < a < 0.9$
10. $0 < a < 1$

Project Reference

In the EF5-2 project "Data-driven Modeling of the Romanization Process of Northern Africa," historical data is used to infer networks that represent the spread of culture. Due to the sparsity of the data, many networks provide similar explanations for the spreading process. Therefore, it is crucial to use algorithms that can explore a wide range of possible solutions. One such algorithm is Particle Swarm Optimization (PSO), which simulates the movement of a bird swarm, with the positions and velocities of the particles governed by a difference equation, like the one shown here. Different parameters can lead to different dynamic behaviors of the algorithm, and these variations can be beneficial depending on the specific problem.

Solution

The correct answer is: 10.

First, we note that the value of the two coordinates v_t or x_t are always the same. Thus, we can restrict ourselves to the first coordinate. We will stick to the notation v_t, x_t, which will now refer to real numbers (the first coordinates of the "old" v_t, x_t)
To make our lives easier, we combine the velocity and position equations into a single equation for the positions:

$$x_{t+1} = x_t + v_{t+1} = x_t - 0.5v_t - ax_t \overset{*}{=} x_t - 0.5(x_t - x_{t-1}) - ax_t,$$

where we used the shifted and rearranged version of the formula for velocity $v_t = x_t - x_{t-1}$ in the equality $*$. Now, we can simplify the terms:

$$x_{t+1} = (0.5 - a)x_t + 0.5x_{t-1}, \quad (t \geq 1).$$

To ensure that the reindeer become and remain arbitrarily slow, the terms in this iteration must decrease over time. To show this, in claim 1 we show that v_t becoming and remaining arbitrarily small is equivalent to x_t becoming and remaining arbitrarily small. In claim 2 we then show, that x_t becomes and remains arbitrarily small, if $0 < a < 1$. This already yields the correct possible answer to this challenge. For those interested, in claim 3 we also prove, that v_t does not become arbitrarily small, if $a \geq 1$:

- **Claim 1:** The reindeer become and remain arbitrarily slow if and only if the position x_t becomes and remains arbitrarily small.
- **Claim 2:** If $0 < a < 1$, then $|x_{2n}| \leq r^n,\ |x_{2n+1}| \leq r^n$ for some fixed $r < 1$ and for all natural $n \geq 1$. (This really ensures that the x_t's become and remain arbitrarily small over time.)
- **Claim 3:** If $a \geq 1$, then for the velocity it holds $|v_t| \geq \min(|v_1|, |v_2|)$ for all $t \geq 2$. Here $\min(a, b)$ denotes the smaller value of a, b.

Claim 1:
Let us first give an intuitive explanation. Suppose that the v_t's become and remain arbitrarily small but the x_t's don't. We now try to find a contradiction. By the definition of v_{t+1} we have

$$v_{t+1} = -0.5v_t - ax_t.$$

If we pick a time t for which v_t and all succinct velocities are negligibly smaller than ax_t (which we can do, as x_t and therefore ax_t doesn't stay arbitrarily small), we can approximate v_{t+1} with

$$v_{t+1} \approx -ax_t.$$

But then v_{t+1} is about as big as ax_t which is a contradiction. In a similar way, it can be seen, that if the x_t's become and stay arbitrarily small, then also the v_t's do.

To make this argument more rigorous, we can choose an arbitrary small number $\epsilon > 0$ and a $T > 1$, such that for all $t > T$, $|v_t| < \epsilon$. By the definition of the x_t's, they also get and stay arbitrarily close to a number x. If x was not 0, then we can achieve a contradiction with the following calculations:

$$\begin{aligned} |x_t - x| < \epsilon \quad &\implies \quad ||x_t| - |x|| < \epsilon \\ &\implies \quad -\epsilon < |x_t| - |x| < \epsilon \\ &\implies |x| - \epsilon < |x_t| \end{aligned}$$

With the definition of v_{t+1} we get:

$$\begin{aligned} |v_{t+1}| &= |-0.5v_t - ax_t| \\ &\geq ||ax_t| - |0.5v_t|| \geq |ax_t| - |0.5v_t| \\ &> |ax_t| - 0.5\epsilon \\ &> |ax| - a\epsilon - 0.5\epsilon. \end{aligned}$$

If we choose ϵ small enough, then the last expression is greater than 0, which means that v_{t+1} cannot become arbitrary small. This is the promised contradiction. To show that if the x_t's become and stay arbitrarily small then also the v_t's do can be proven similarly. □

Claim 2:
We first calculate the first two iterations of the position:

$$\begin{aligned} x_0 &= 1, \\ x_1 &= 1 - a. \end{aligned}$$

Therefore, $|x_1| < |x_0|$ if and only if $0 < a < 2$.
For the next position, we can apply the triangle inequality (see hints and remarks):

$$\begin{aligned} |x_2| &= |(0.5 - a)x_1 + 0.5x_0| \\ &\leq |(0.5 - a)x_1| + |0.5x_0| \\ &= |(0.5 - a)||x_0| + |0.5x_0| \\ &< 0.5 + 0.5 = 1, \end{aligned}$$

where we used that $|0.5 - a| < 0.5$ for $0 < a < 1$. Therefore, there exists such an r, and we can define it as $r := |(0.5 - a)| + 0.5$. Thus, $|x_2| < |x_0| \cdot r = r$.
It follows for x_3:

$$\begin{aligned} |x_3| &= |(0.5 - a)x_2 + 0.5x_1| \\ &\leq |(0.5 - a)x_2| + |0.5x_1| \\ &\leq |(0.5 - a)x_0| + |0.5x_0| = r|x_0| = r. \end{aligned}$$

And in general, we can prove claim 2 in the following way: Suppose we know that claim 2 is true up to a number n. We than want to show that it is true for $n + 1$. Because for $n = 1$ we already know that this is true, we could then conclude that the statement is true for all natural numbers n. This reasoning is known as proof by induction. It holds:

$$\begin{aligned} |x_{2n+2}| &= |(0.5 - a)x_{2n+1} + 0.5x_{2n}| \\ &\leq |0.5 - a||x_{2n+1}| + 0.5|x_{2n}| \\ &\leq |0.5 - a|r^n + 0.5r^n = r^{n+1}, \end{aligned}$$

and

$$\begin{aligned} |x_{2n+3}| &= |(0.5-a)x_{2n+2} + 0.5x_{2n+1}| \\ &\leq |0.5-a|r^{n+1} + 0.5r^n \\ &< |0.5-a|r^n + 0.5r^n = r^{n+1}. \end{aligned}$$

where we used $r < 1$. As $r < 1$, the position becomes and stays arbitrary small. □

Claim 3:
Now, we show that if $a > 1$, the velocity is always larger than the absolute value of the velocity at the second or first time step and therefore cannot become arbitrarily small.
Before we do that, note, that there is a formula for the v_t's only dependent on the velocity, similar to the formula for the position:

$$v_{t+1} = -0.5v_t - ax_t \overset{*}{=} (0.5-a)v_t + 0.5v_{t-1},$$

where we used in $*$ the iterative definition of $v_t = -0.5v_t - ax_t$ to remove the $-ax_t$ term.

Consider the following observations:

$$v_1 = -ax_0 = -a < 0, \quad (a > 0)$$

$$v_2 = -0.5v_1 - ax_1 = -0.5(-a) - a(1-a) = a^2 - 0.5a > 0 \quad (a > 0.5).$$

In general, with induction, we can prove, that the v_t's have alternating signs, with $v_{2n} > 0$ and $v_{2n+1} < 0$. We have:

$$\begin{aligned} v_{2(n+1)} = v_{2n+2} &= \underbrace{(0.5-a)}_{<0}\underbrace{v_{2n+1}}_{<0} + \underbrace{0.5v_{2n}}_{>0} > 0 \\ v_{2(n+1)+1} = v_{2n+3} &= \underbrace{(0.5-a)}_{<0}\underbrace{v_{2n+2}}_{>0} + \underbrace{0.5v_{2n+1}}_{<0} < 0. \end{aligned}$$

With this knowledge at hand, we can finally prove the claim, again by induction:

$$\begin{aligned} |v_{2n+2}| = v_{2n+2} &= (0.5-a)\underbrace{v_{2n+1}}_{<0} + 0.5v_{2n} \\ &= \underbrace{|0.5-a|}_{\geq 0.5, \text{ if } a \geq 1} |v_{2n+1}| + 0.5|v_{2n}| \\ &\geq 0.5|v_{2n+1}| + 0.5|v_{2n}| \\ &\geq \min(|v_{2n+1}|, |v_{2n}|) \geq \min(|v_1|, |v_2|), \end{aligned}$$

$$\begin{aligned} |v_{2n+3}| = -v_{2n+3} &= -(0.5-a)v_{2n+2} - 0.5v_{2n+1} \\ &= \underbrace{|0.5-a|}_{\geq 0.5, \text{ if } a \geq 1} v_{2n+2} + 0.5|v_{2n+1}| \\ &\geq 0.5|v_{2n+2}| + 0.5|v_{2n+1}| \\ &\geq \min(|v_{2n+2}|, |v_{2n+1}|) \geq \min(|v_1|, |v_2|). \end{aligned}$$

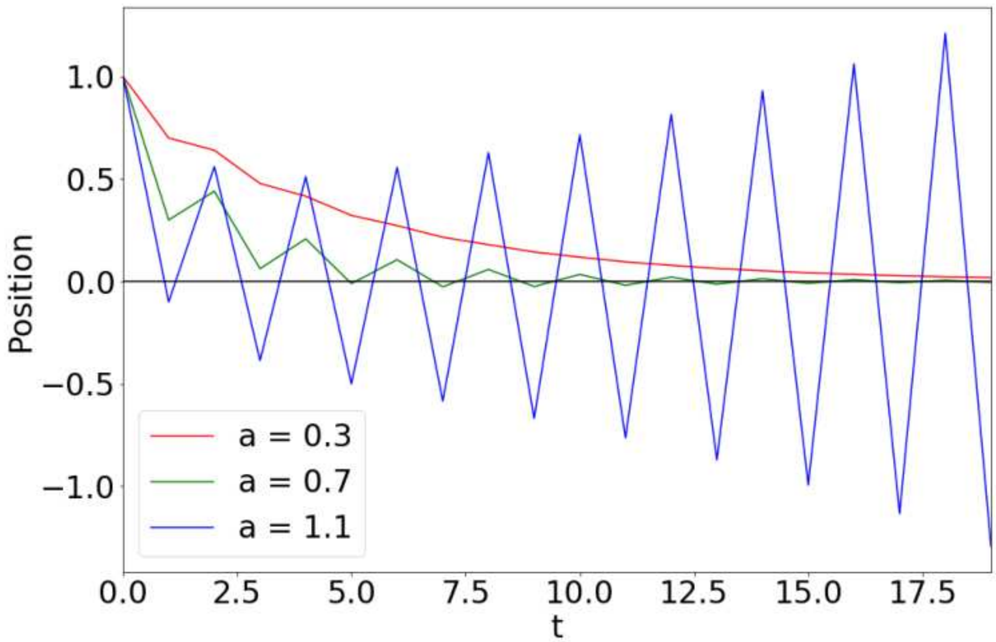


Figure 14: Position of the First Component over Time for Different a Values

□

Thus, the reindeer will become and stay arbitrarily slow if and only if $0 < a < 1$.

In Fig. 14, we illustrated the movement of one component of the reindeer with $a = 0.3$, 0.7, and 1.1. As shown, the reindeer with $a = 0.3$ and $a = 0.7$, depicted in red and green, respectively, move towards position 0, while the reindeer with $a = 1.1$, shown in blue, accelerates and exhibits oscillatory behavior.

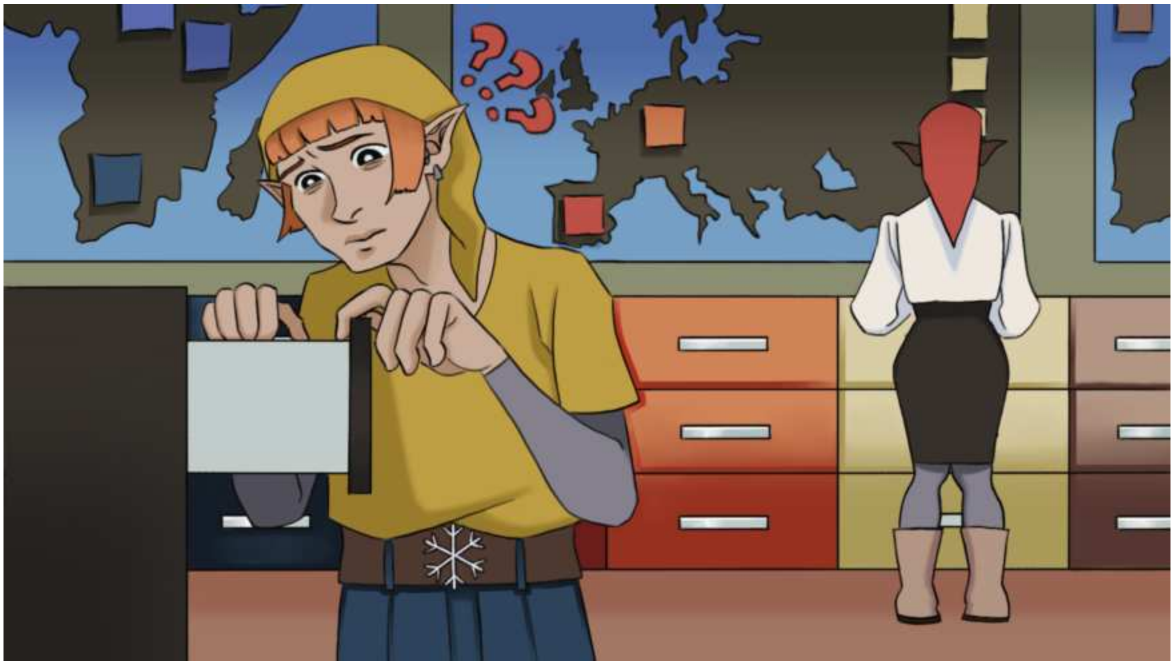


Illustration: Julia Nurit Schönnagel

9 Christmas Logistics

Authors: Marieke Heidema & Amir Shakouri (University of Groningen)

Challenge

Every year, Santa's elves keep track of the number of people that live in a place. That way, they know exactly how many presents to make for each town and city. They write down this number in a book. This book also contains other information about the place, like the location and the weather forecast for that place.

When they check the book shortly before Christmas, they cannot find the population of a small island in the Atlantic. The elves start to panic. "What do we do now," they ask each other. A hasty discussion takes place. "Can we not just quickly go to the island and count the number of inhabitants again?" asks young Beatrice. "No time, no time!" answers Theo. Concerned, elf Leonard asks "So then we have to guess how many presents we need, but what if we don't bring enough presents?" Matthew, one of the senior elves, adds to this "Well, we can bring some extra presents, but we can surely not bring too many presents. That will make Santa's sled too heavy to pull for the reindeers. It will slow the sled down and the presents may not be delivered on time!" "So what can we do then?" Beatrice asks, "We cannot bring too few, nor way too many. So how many do we bring?"

The elves fall silent, they think and look around at each other, hopeful. But no one seems to know what to do. Then, a soft cough comes from the doorstep, it's Lennard, a young elf. It's his first year helping with the Christmas preparations and he stayed quiet until now. When he coughs, the other elves look around at him. “Is there anything you want to say, Lennard? Do you have an idea?” Theo asks. In a soft but confident voice, Lennard answers “Yes, I think I know what to do. Let me explain...”

Lennard points to the page in the book with the island’s data. There is an old note about the island there, which reads as follows:

Christmas 2022 : ***Attention, new island, don’t forget to count the inhabitants!***
Note, there is an ongoing project aiming to create new spaces for people to live. Islands that were not inhabited before are now being inhabited. This particular island has space for 200 people but is initially only inhabited by 25 people. The increase rate with which the population will grow is thought to be 2.

Lennard pushes his round glasses further on his nose and starts to explain: “Look, we can use mathematics to estimate how many people live on the island now.” He writes the following on a blackboard:

$$P(t+1) - P(t) = \left(r - \frac{r}{C}P(t)\right) P(t).$$

“Here, $P(t)$ is the size of the population in year t. The note tells us that the initial population size in 2022 was 25, meaning that $P(2022) = 25$. Furthermore, r stands for the approximate rate of increase of the population. By the note we know that $r = 2$. Lastly, C is the capacity of the island, which we know by the note is $C = 200$.”

Based on this, what will be the estimated number of inhabitants in Christmas 2024?

Possible Answers:

1. 0 inhabitants.
2. 1 to 22 inhabitants.
3. 23 to 55 inhabitants.
4. 56 to 71 inhabitants.
5. 72 to 98 inhabitants.
6. 99 to 123 inhabitants.
7. 124 to 150 inhabitants.
8. 151 to 183 inhabitants.
9. 184 to 199 inhabitants.
10. 200 inhabitants.

Solution

The correct answer is: 8.

From the challenge we know the parameters

$$r = 2 \quad \text{and} \quad \tfrac{r}{C} = \tfrac{2}{200} = 0.01$$

and the initial value $P(2022) = 25$. Therefore, we can calculate the population in 2023, using what we know about the population size in 2022 as follows:

$$P(2023) = (2 - 0.01 \cdot P(2022)) \cdot P(2022) + P(2022) = 68.75.$$

Similarly, the population of 2024 can be estimated using the estimated population in 2023 in the following way:

$$P(2024) = (2 - 0.01 \cdot P(2023)) \cdot P(2023) + P(2023) \approx 158.98$$

Hence, the estimated population size in 2024 belongs to the interval in answer 8.

Remark:

The equation

$$P(t+1) - P(t) = \left(r - \frac{r}{C}P(t)\right) P(t)$$

models the population $P(t)$ each year. This discrete-time equation can also be approximated by a continuous-time differential equation, called *the logistic equation*:

$$\frac{\mathrm{d}}{\mathrm{d}t}P(t) = \left(r - \frac{r}{C}P(t)\right) P(t). \tag{3}$$

Here, $P(t)$ represents the population size at any moment, and the differential equation captures how the population evolves continuously over time.
Combining the initial condition with the differential equation, we have the following *initial value problem*:

$$\tfrac{\mathrm{d}P(t)}{\mathrm{d}t} = (2 - 0.01P(t))\, P(t), \quad P(2022) = 25.$$

Here, note that we can approximate the derivative of P at t as

$$\tfrac{\mathrm{d}P(t)}{\mathrm{d}t} \approx \tfrac{P(t+\Delta t) - P(t)}{\Delta t}$$

when Δt is small enough. For example, if we take $\Delta t = 1$ year, then we arrive exactly at the time-discrete equation from Lennard. Discretization is a common technique in numerical mathematics. It is often used to approximate complex initial value problems that cannot be solved analytically, by breaking them into discrete steps. Using *separation of variables*, we can rewrite the differential equation (3), as the following integral equation:

$$\int \frac{1}{(r - \frac{r}{C}P(t))P(t)} \mathrm{d}P(t) = \int \mathrm{d}t.$$

Computing the integrals and taking the initial condition $P(2022) = 25$ into account, one can find that the solution to the initial value problem is given by

$$P(t) = \frac{CP(2022)}{(C - P(2022))e^{r(2022-t)} + P(2022)}.$$

Plugging in the values $r = 2$, $C = 200$ and $P(2022) = 25$, we can now compute what the population size will be in 2024:

$$P(2024) = 177.272 \approx 177.$$

Note that this population size is again in the interval of answer 8.

Illustration: Ivana Martić

10 Cutting Christmas Cookies

Authors: Pim van 't Hof, Stefano Piceghello (University of Twente)
Project: 4TU.AMI

Challenge

Pixies Pi and Pie are baking perfectly round Christmas cookies. Using a circular cookie cutter, they cut out four identical cookies from a large circular piece of dough. Figure 15 shows the piece of dough that remains after the cookies have been put in the oven. The figure also shows two perpendicular diameters AB and CD of the large circle. Diameter AB is tangent to the boundary of one of the circular holes. Line segment EF is a chord of the large circle that is tangent to the boundary of the same hole and parallel to diameter AB. The length of EF is 36 cm.

What is the area of the remaining piece of dough?

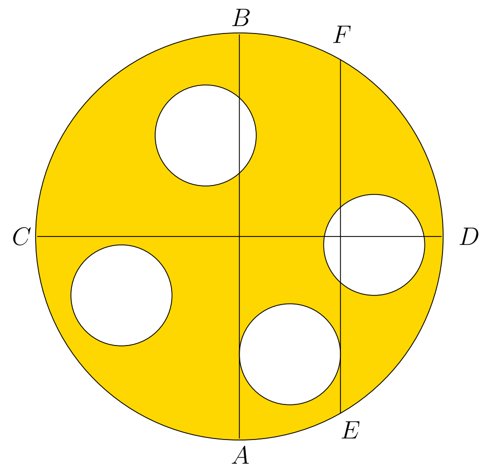


Figure 15: The remaining piece of dough.

Possible Answers:

1. 244π cm^2
2. 300π cm^2
3. 312π cm^2
4. 320π cm^2
5. 324π cm^2
6. 344π cm^2
7. 360π cm^2
8. 368π cm^2
9. 381π cm^2
10. There is not enough information in the problem statement to compute the area of the remaining piece of dough.

Solution

The correct answer is: 5.

We consider only the right bottom quarter of the large piece of dough and the hole whose boundary is tangent to line segments AB and EF in Figure 15. If we denote by O the center of the large circle and by S the intersection point of the line segments CD and EF, we get the situation depicted in Figure 16. We will compute the shaded area in Figure 16, which clearly equals $\frac{1}{4}$ of the area we are looking for, i.e., $\frac{1}{4}$ of the shaded area in Figure 15.

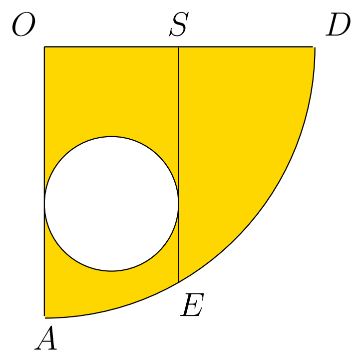


Figure 16: The problem reduced to a simpler one.

Let R denote the radius of the large circle and d denote the diameter of the circular hole. Then, in Figure 16, $OD = OA = OE = R$. We know from the description of the problem that $EF = 36$ cm in Figure 15 and that EF is perpendicular to diameter AB, so $ES = 18$ cm. By the Pythagorean theorem, it follows that

$$OS = \sqrt{(OE)^2 - (ES)^2} = \sqrt{R^2 - 18^2} \text{ cm.}$$

Since $OS = d$ is the diameter of the circular hole, the area of the circular hole equals

$$\pi \left(\frac{1}{2}\sqrt{R^2 - 18^2}\right)^2 = \frac{1}{4}\pi \left(R^2 - 18^2\right) \text{ cm.}$$

The area of the large quarter circle before the hole was cut out was equal to $\frac{1}{4}\pi r^2$ cm^2, so the shaded area in Figure 16 is equal to

$$\frac{1}{4}\pi R^2 - \frac{1}{4}\pi \left(R^2 - 18^2\right) = \frac{324}{4}\pi \text{ cm}^2.$$

We conclude that the shaded area in Figure 15, i.e., the area of the remaining piece of dough after pixies Pi and Pie cut out the four cookies, equals 324π cm^2.

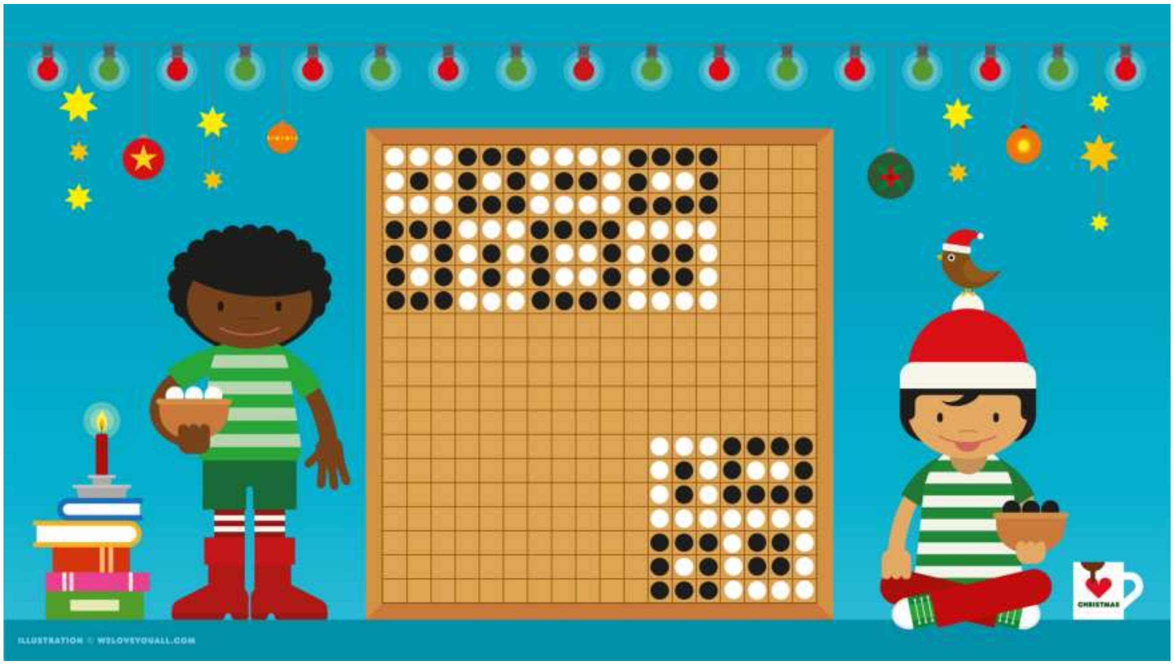


Illustration: Friederike Hofmann

11 Go for Christmas!

Author: Hajo Broersma
Project: 4TU.AMI

Challenge

In order to pass the time on a dark rainy day before Christmas, the three dwarfs Godric, Goliath and Gowan take up the challenge to fill up the 18×18 squares of a traditional Go board with white and black stones in an artistic and visually attractive way. To reach their goal they agree to use the following basic blocks for their designs:

1. Bricks consist of a single-colored strip exactly one row wide, surrounded by a frame of the other color. The notation k-brick for $k \geq 1$ describes a brick whose single-colored strip has a length of k.

2. Tiles consist of a single-colored square in the center, surrounded by a frame of the other color. The notation k-tile for $k \geq 1$ describes a tile whose central single-colored square has a size of $k \times k$.

For example, a white 1-brick, a white 2-brick, and a black 4-brick are shown on the left in Figure 17. Similarly, a black 1-tile, a black 2-tile, and a white 4-tile are shown on the right in Figure 17.
A 1-brick and a 1-tile of the same color are therefore identical.

The challenge of the three dwarfs is to use the basic blocks to fill up the 18×18 squares of a traditional Go board with white and black stones in such a way that:

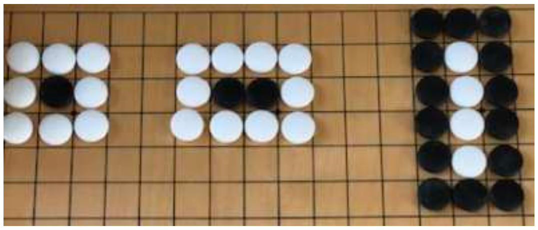
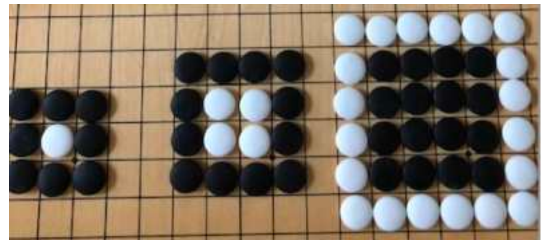



Figure 17: Representation of bricks and tiles. Left: A white 1-brick, a white 2-brick, and a black 4-brick. Right: A black 1-tile, a black 2-tile, and a white 4-tile.

- all the squares are covered;
- the total number of used white stones is equal to the total number of used black stones;
- Two different basic blocks with the same frame color do not touch, but maybe adjacent at the corners.

Figure 18 shows a partial cover that is allowed (left upper corner) and a partial cover that is not allowed (right lower corner; the red dots indicate a conflict).

Figure 18: An allowed (upper left) and a disallowed (lower right) partial cover

The dwarves now aim to use as many basic blocks of different sizes as possible. They refer to the number of basic blocks of varying sizes as the aesthetic quality. Colors are ignored for this count. As an example: The partial covering in the upper part of Figure 18 uses 1-bricks (1-tiles), 2-bricks, 2-tiles, and no other basic blocks. Therefore, this partial covering of a 7×14 board would receive a score of 3 for aesthetic quality.
The three dwarfs examine different strategies for designing proper covers. Hereby,

- Godric is only allowed to use 1-bricks, 2-bricks, 3-bricks etc., but no k-tiles with $k \geq 2$
- Goliath is only allowed to use 1-tiles, 2-tiles, 3-tiles etc., but no k-bricks with $k \geq 2$

- Gowan is free to use a mixture of any of the bricks and tiles.

Each of them tries to obtain the highest possible score, respecting all the restrictions.
Question: What are the highest possible scores of Godric, Goliath and Gowan?

Possible Answers:

1. The highest scores of Godric, Goliath and Gowan are 3, 1 and 3, respectively.
2. The highest scores of Godric, Goliath and Gowan are 3, 1 and 4, respectively.
3. The highest scores of Godric, Goliath and Gowan are 3, 2 and 5, respectively.
4. The highest scores of Godric, Goliath and Gowan are 4, 1 and 4, respectively.
5. The highest scores of Godric, Goliath and Gowan are 4, 1 and 5, respectively.
6. The highest scores of Godric, Goliath and Gowan are 4, 2 and 6, respectively.
7. The highest scores of Godric, Goliath and Gowan are 5, 1 and 5, respectively.
8. The highest scores of Godric, Goliath and Gowan are 5, 2 and 5, respectively.
9. The highest scores of Godric, Goliath and Gowan are 5, 2 and 6, respectively.
10. None of the other possible answers is correct.

Solution

The correct answer is: 4.

There exists a proper cover satisfying all of the restrictions by simply using only 1-bricks (1-tiles)

This is the case since the sides of a 1-brick (1-tile) contain exactly 3 stones, so each of the six 3×18 columns of an 18×18 Go board can be properly covered by six 1-bricks (1-tiles) that alternate in colors, and neighboring columns can get alternating colors as well. Altogether this cover consists of 36 1-bricks (1-tiles), of which 18 are white and 18 are black, so the cover is proper. Since all the used basic blocks in this cover are the same (except for the alternating colors, which we should ignore), this cover gets a score of 1.

By using k-tiles only it is not possible cover properly using different tiles

We are next going to show that Goliath cannot find a proper cover using more than one type of tiles, so Goliath's maximum score is 1. Suppose now that Goliath starts to cover the left upper corner of the Go board with a k-tile and next to it with a ℓ-tile for $\ell \geq k$. Putting two tiles of different sizes and alternating colors next to each other creates a corner with one black stone across one edge and one white stone across another edge. It is therefore not possible to put any stone in the corner.

Cover with bricks can obtain score of 4

First, we show one possible cover with 4 different sized bricks in Figure 19.

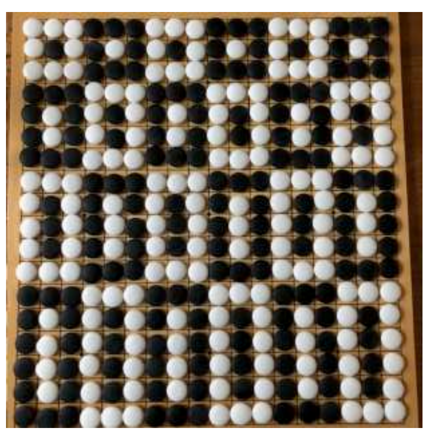


Figure 19: A proper cover with score 4

Cover with bricks can obtain score of maximal 4

Suppose there is a proper cover with a score of 5 or higher. Then this cover uses at least k-bricks with five different values of $k \geq 1$. Apart from the 1-bricks, these non-square k-bricks together have long sides summing up to at least $4 + 5 + 6 + 7 = 22$ stones, each having a short side consisting of 3 stones. Recall that along the long side of a k-brick with $k \geq 2$ we can only put k-bricks with the same fixed value of k, whereas along the short side we can put any ℓ-brick, like in the constellation in Figure 19. Since the Go board is 18×18, we cannot put 22 or more stones on the left side (the first column) or the upper side (the first row) of the board, so we will be forced to put at least one long side of a non-square k-brick on either of these sides. But then we encounter a conflict at the intersection of the rows and columns corresponding to these long sides, as different k-bricks meet there. The bricks placed at this intersection cannot have a side consisting of 3 stones due to the orientation of the different k-bricks. Otherwise, a corner would form, which, as previously shown, is not allowed.

Cover with bricks and tiles can obtain a maximal score of 4

Let us now argue why the use of an additional k-tile with a fixed value $k \geq 2$ will not lead to a higher score. This is due to the fact that next to such a k-tile we can only put either another k-tile with opposite colors or a k-brick (with opposite colors) along the long side of the k-brick. Repeating the recursion, the whole $(k + 2) \times 18$ row and column of the Go board containing the k-tile must be filled up with k-bricks and k-tiles only. Due to the corner problem we mentioned before, next to a k-brick one can only put another k-brick (on the long side or the short side), a k-tile (on the long side) or a 1-brick. Therefore, such a cover cannot reach a higher score than 3.

The above arguments imply that a score above 3 is only possible if no k-tiles with $k \geq 2$ are used in the cover, so we can assume that only k-bricks are used (since a 1-tile is a 1-brick). We already showed such a proper cover with a score of 4, so we deduce that the highest scores of Godric and Gowan are both 4.

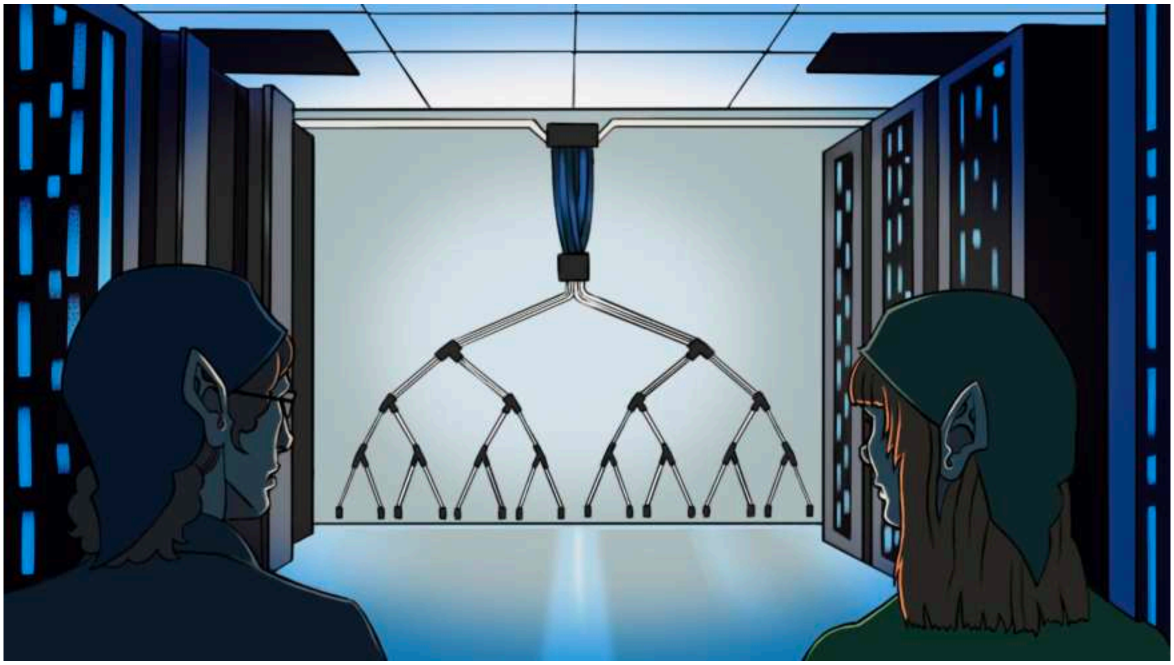


Illustration: Julia Nurit Schönnagel

12 Cable Chaos

Author: Lukas Protz
Project: MATH+

Challenge

Paula and Quentin are the head elves of the IT department at the North Pole. Usually, everything in the department goes according to plan, but one day they receive messages from two of the elves working there. They complain about having issues connecting to the winternet (the analog of the internet at the North Pole).

As the connection to the winternet is done with cables, Paula and Quentin think that some of the cables might be damaged, so they go to the cable room checking them. The cable terminals that correspond to the two elves are labeled "$\frac{22}{7}$" and "$\frac{87}{32}$". Note that, despite the labels of the cables being in the form of fractions, they are just labels and (a priori) have no connection to rational numbers.

Once inside the cable room, Paula and Quentin get overwhelmed by a huge cable harness. How on earth should they find the two cable terminals they are searching for? Luckily, they find a note and a picture (compare Figure 20) on the wall explaining the labeling system of the cable terminals:

(*Figure on next page*)

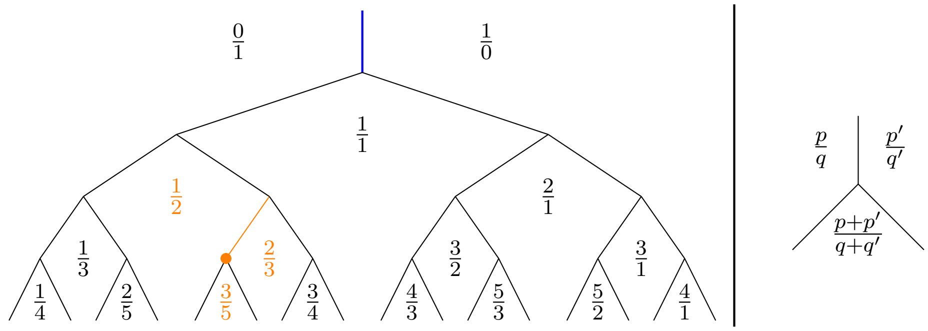


Figure 20: **Left:** The roots of the tree with the corresponding labels. **Right:** The rule to obtain a new label if a root splits and one cable ends. The orange dot denotes the end of the cable labeled "$\frac{3}{5}$".

1. The cables are arranged in a tree structure (the harness), see Figure 20. The cable bundles are the individual roots (edges) of the harness. There is one main root of the tree, colored blue in Figure 20.
2. The region to the left of the main root is labeled "$\frac{0}{1}$", and the region to the right is labeled "$\frac{1}{0}$".
3. Each time a root splits into two new roots, one cable ends and the others continue along one of the two new roots.
4. The main root splits into two new roots. The cable that ends at this splitting point has the label "$\frac{1}{1}$", which is obtained by combining the labels of the left and right region in the following way:
$$\frac{0+1}{1+0} = \frac{1}{1}.$$
The newly created region by the two new roots is also labeled "$\frac{1}{1}$".
5. This process continues indefinitely. Each root splits into two new roots, one cable ends and a new region is created. If the labels of the right and left region of the root before the splitting point are "$\frac{p}{q}$" and "$\frac{p'}{q'}$", then the cable that ends and the new region are labeled "$\frac{p''}{q''}$", according to the following rule:
$$\frac{p''}{q''} = \frac{p+p'}{q+q'}.$$
6. For example, the labels of the regions to the left and right of the root highlighted in orange in Figure 20 are "$\frac{1}{2}$" and "$\frac{2}{3}$" and the label of the new region and ending cable at the orange dot is
$$\frac{1+2}{2+3} = \frac{3}{5}.$$

Now to find the cables in question, Paula and Quentin start from the main root. Each time a root splits, they continue following either the left or the right roots until they arrive at a splitting point for one of the searched cables. Can you help Paula and Quentin to find the ends of the cables labeled "$\frac{22}{7}$" and "$\frac{87}{32}$" by determining the sum of right turns of Paula and Quentin and the sum of left turns of Paula and Quentin?

Example: To get from the main root to the splitting point where the cable labeled "$\frac{3}{5}$" is, they first have to turn left, then right, and finally left again.

Remark: To clarify our terminology around left and right regions, we define these concepts with respect to the main root. Define the distance of a splitting point from the main root as the number of splitting points along the path from the main root to that splitting point, moving through intermediate roots in the harness. For example, the splitting point labeled "$\frac{1}{2}$" has a distance of 2 from the main root.
A splitting point is said to be closer to the main root than another if its distance to the main root is less than that of the other point.
Each root connects two splitting points, one of which is closer to the main root. When traversing from the splitting point that is farther from the main root to the one that is closer, the left region is defined as the region in the direction of a 90° counterclockwise rotation from the traversal direction, while the right region is in the direction of a 90° clockwise rotation. The left root now is the new root at a splitting point that touches the left region. Similarly, the right root is the new root that touches the right region.

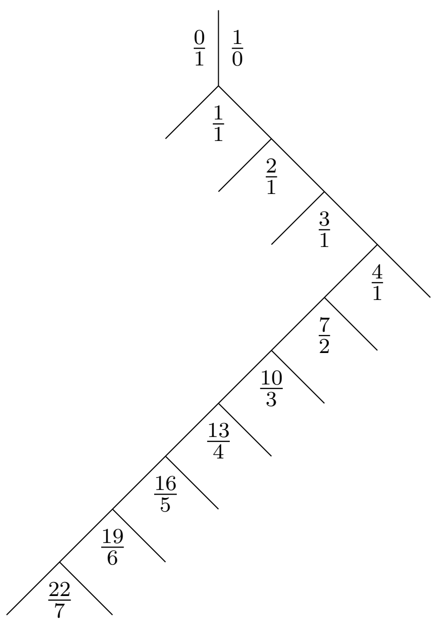
Possible Answers:

1. 6 left and 8 right turns
2. 11 left and 8 right turns
3. 8 left and 9 right turns
4. 10 left and 9 right turns
5. 7 left and 10 right turns
6. 9 left and 10 right turns
7. 8 left and 11 right turns
8. 7 left and 12 right turns
9. 6 left and 13 right turns
10. Not both labels appear in the tree.

Solution

The correct answer is: 2.

For simplicity, we will drop the quotation marks around the labels.
Of course, one way of solving this challenge is to simply search the tree with respect to the length of a path. Eventually, one then finds the correct path. However, there are some observations that can dramatically reduce the amount of computations needed. We will discuss them further down. At first, let us confirm, that the solution is right. To get to the splitting point where the cable labeled $\frac{22}{7}$ ends, one has to turn 3 times right and then 6 times left. To get to $\frac{87}{32}$ one has to turn 2 times right, then left, then 2 times right, then left, then right and finally 3 times left. Indeed, Figure 21 shows that the numbers appear at the claimed positions and the sum of right turns is 8, while the sum of left turns is 11.



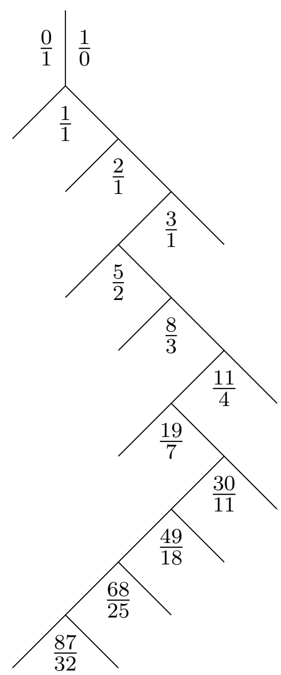


Figure 21: **Left:** The path to $\frac{22}{7}$. **Right:** The path to $\frac{87}{32}$.

To reduce the amount of computations to find the above solution, the following observations can be made:

The labels of the regions and cables can be directly identified with fractions, with the only exception being $\frac{1}{0}$. We will also interpret this expression as some kind of number in the following way: For every fraction $\frac{a}{b}$, with a being an integer and b being a positive integer we declare

$$\frac{a}{b} < \frac{1}{0}.$$

Consider a fraction in the tree $\frac{p}{q}$ with p being an integer and q being a positive integer obtained from the fractions $\frac{a}{b}$ and $\frac{c}{d}$:

$$\frac{p}{q} = \frac{a+c}{b+d},$$

with a, $b > 0$, c, $d > 0$ satisfying

$$\frac{a}{b} < \frac{c}{d}.$$

We will show that $\frac{p}{q}$ lies between $\frac{a}{b}$ and $\frac{c}{d}$, i.e.

$$\frac{a}{b} < \frac{p}{q} < \frac{c}{d}.$$

First, we prove

$$\frac{a}{b} < \frac{a+c}{b+d},$$

or equivalently that it holds

$$a(b+d) < (a+c)b.$$

To show this, note that per construction

$$\frac{a}{b} < \frac{c}{d},$$

which is equivalent to

$$ad < bc.$$

Thus,

$$a(b+d) = ab + ad < ab + bc = (a+c)b.$$

Similarly, we can show that

$$\frac{a+c}{b+d} < \frac{c}{d}$$

by manipulating the inequality:

$$(a+c)d = ad + cd < bc + cd = c(b+d).$$

This completes the proof.

This sequence of inequalities remains true if we choose $\frac{c}{d} = \frac{1}{0}$:

$$\frac{a}{b} < \frac{a+1}{b+0} < \frac{1}{0}.$$

With this observation we can search the tree much more efficiently in the following way: The label we need, let us call it l, always has to be between the two fractions $\frac{a}{b}$ and $\frac{c}{d}$ in the left and in the right region. When the root splits we obtain a new fraction $\frac{p}{q}$ in between the other two fractions.

Case 1 $\frac{p}{q} = l$: If this is the case, we are done.
Case 2 $\frac{a}{b} < l < \frac{p}{q}$: In this case we have to follow that root with $\frac{a}{b}$ and $\frac{p}{q}$ in its left and right region.
Case 3 $\frac{p}{q} < l < \frac{c}{d}$: In this case we have to follow that root with $\frac{c}{d}$ and $\frac{p}{q}$ in its left and right region.

Alternative Solution

There is another very elegant way to compute the path to a fraction in the tree, which involves the notion of a continued fraction. However, we will focus only on how to obtain a fraction in the tree using continued fractions. We will not delve into the reasons why these advanced mathematical tools are necessary; instead, we encourage interested readers to explore the terms "mediant" and "Stern-Brocot tree" online for further understanding.

A continued fraction with denominators $a_0, a_1 \dots, a_n$ from the natural numbers is given by the following expression:

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}.$$

For example $\frac{22}{7}$ is represented by the continued fraction

$$\frac{22}{7} = 3 + \frac{1}{7},$$

and $\frac{87}{32}$ is represented by

$$\frac{87}{32} = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4}}}}}.$$

We will show how to obtain the continued fraction of a rational number by considering the example of $\frac{87}{32}$. The key step is to divide the numerator by the denominator and keep the rest.

$$\begin{aligned}
\frac{87}{32} &= 2 + \frac{23}{32} &&= 2 + \cfrac{1}{\frac{32}{23}} \\
&= 2 + \cfrac{1}{1 + \frac{9}{23}} &&= 2 + \cfrac{1}{1 + \cfrac{1}{\frac{23}{9}}} \\
&= 2 + \cfrac{1}{1 + \cfrac{1}{2 + \frac{5}{9}}} &&= 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{\frac{9}{5}}}} \\
&= \dots &&= 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4}}}}}
\end{aligned}$$

Now we can use the continued fraction to find a path to the corresponding rational number in a very simple way: If $a_0, a_1, a_2, a_3, \ldots, a_n$ are the denominators of the continued fraction, then one has to turn a_0 times right, then a_1 times left, then a_2 times right, then a_3 times left, ... , and finally $a_n - 1$ times either right, if n is odd or left, if n is even.

For $\frac{22}{7}$ this means one has to turn 3 times right, then $7 - 1$ times left.
For $\frac{87}{32}$ this means one has to turn 2 times right, then 1 time left, then 2 times right, then 1 time left, then 1 time right, then $4 - 1$ times left.

Illustration: Zyanya Santuario

13 Surrounded by Rubies

Author: Matthew Maat (Universiteit Twente)
Project: Combining algorithms for parity games and linear programming

Challenge

Somewhere in the Far East, the wise men are packing their gifts to present to the newborn king. As they are loading them into their camel bags, they realize that one of the cube-shaped boxes, which is decorated with rubies at some of its corners, is broken (Figure 22, left). The ruby marked with A had fallen off. They can spot the hole in the design quickly because of an ancient eastern decoration pattern, where the rubies on the cube-shaped box are always placed in such a way that

- all the rubies are connected by edges
- the corners with no rubies, called non-rubies, are all connected by edges.
- And the same holds for every square (face of the box), i.e. all rubies in a square and all non-rubies are connected via edges.

In the blue square on this broken box, ruby B is now not connected to ruby C, so they see that the box must be broken. While the wise men agree that the old box had a nice way of detecting if it is broken, it is not perfect: for example, if ruby B had fallen off instead of ruby A, you would not be able to detect it using the rules. So instead of repairing the old

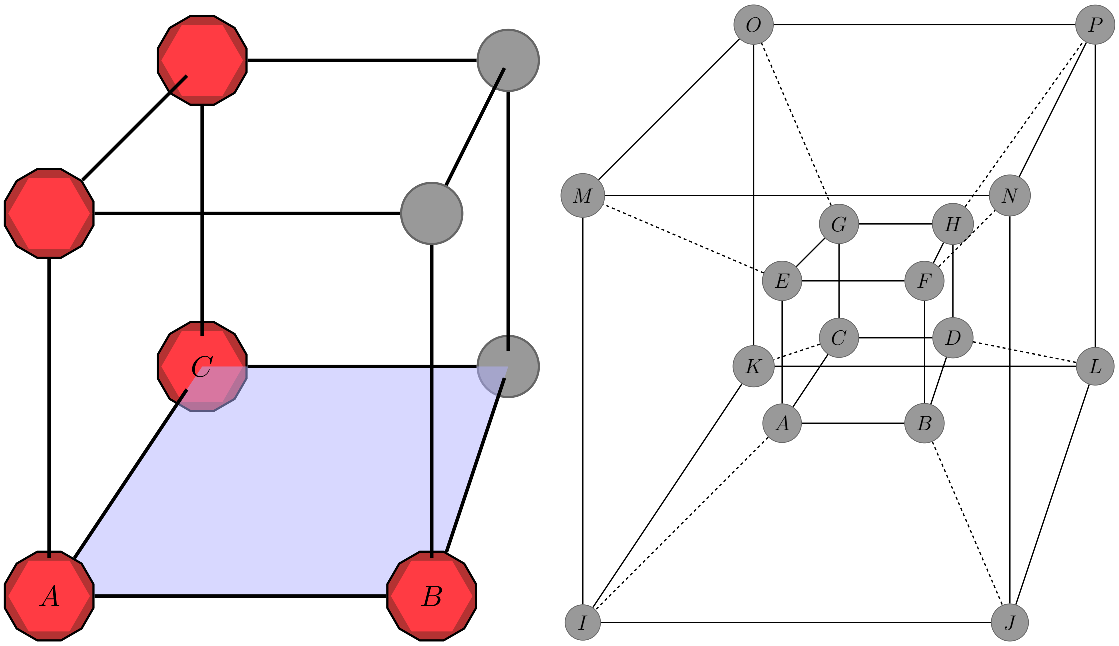


Figure 22: Left: original design of the (now broken) box. The rubies are colored in red. Right: the structure of a tesseract.

box, they want to make a new box. This time they want to use a so-called tesseract (a 4-dimensional hypercube, see description at the end of the challenge). They want some corners of the tesseract to have rubies as well and have the following requirements for the design:

1. There is at least one ruby in the design.
2. All the rubies are connected with each other via some sequence of edges of the tesseract, and all non-ruby corners are also connected with each other by the edges of the tesseract.
3. In every cube in the tesseract, the same holds: all the rubies are connected, and all the non-rubies are connected.
4. In every square, also all the rubies are connected, and the non-rubies are connected
5. You can easily detect if one ruby goes missing: if any ruby disappears, there will be a square in which the rubies are not connected (in the same way as the blue square in Figure 22 left if ruby A disappears). Note, that we also consider the rubies connected, if there is no ruby on a square.

The wise men think for a long time, and find a design that meets not only the requirements, but also has the most rubies among all other designs meeting the requirements.

Question: How many rubies are placed on the tesseract in their design?

About the tesseract: In the same way that you can create a cube by taking two squares and connecting their four corners with edges, you can create a tesseract (in four dimensions) by taking two cubes and connecting the eight pairs of corners of the cubes (see Figure 22 right).

A tesseract has 16 corners ($A, B, C, \ldots, P$), 32 edges (for example, edges AB, JN and CK), 24 squares (like $ABDC$, $IKOM$ or $JBFN$), and 8 cubes (for example, A, B, C, D, E, F, G, H form a cube, and A, B, E, F, I, J, M, N form a cube). Since the tesseract is four-dimensional, the illustration 22 on the right is merely a projection. Consequently, squares or even cubes may appear distorted in the illustration.

Possible Answers:

1. 4 or less rubies
2. 5 rubies
3. 6 rubies
4. 7 rubies
5. 8 rubies
6. 9 rubies
7. 10 rubies
8. 11 rubies
9. 12 rubies
10. 13 or more rubies

Solution

The correct answer is: 5.

If we ignore rotation and reflection, there is only one design possible that meets the requirements. It has 8 rubies, as you can see in Figure 23. The rubies are connected in every square, cube, and in the entire tesseract, and the same holds for the non-rubies. Also, removing any of the rubies will disconnect the rubies in some square.

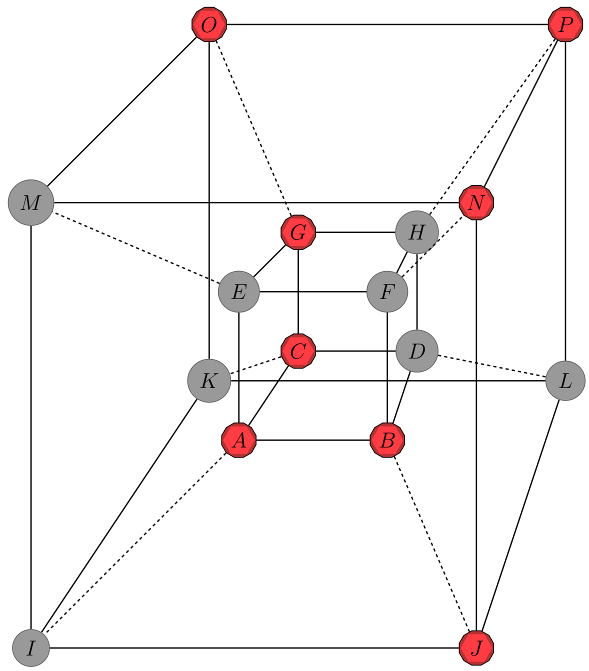


Figure 23: One possible design

Now follows a proof that this design is the only solution. We first look, for some fixed single ruby, how many rubies it can be directly connected to with an edge. It will turn out that any ruby has to be connected to exactly two other rubies. We will therefore disprove all the other cases.

- If some ruby is connected to 0 other rubies, it must be the only ruby, since all rubies have to be connected. But then removing it does not disconnect the set of rubies (as it becomes the empty set).
- If some ruby is connected to one other ruby, we can remove it without disconnecting the other rubies in any square.
- Suppose some ruby is connected to three or more other rubies. By symmetry of the tesseract, we can assume that ruby A is connected to rubies B, C, E (see Figure 24). We can now say a lot about what the design must look like. By rule 5, there must be a square where some rubies get disconnected if A is removed, i.e. there is a square where the corners that are connected to A via an edge have rubies, but not the remaining corner in this square. Therefore again, by symmetry of the tesseract we may assume D to have no ruby. We also know that removing B disconnects the rubies in some square,

by rule 5, so there must be a square where there is no ruby opposite to B and there are two rubies connected to B. Since there is a ruby at E, this cannot be the square $ABFE$, and also there is no ruby at D, so it must be either square $ABJI$ or $BJNF$. In both cases, there must be a ruby at J. With the same logic for when ruby C is removed, there must be a ruby at K. Then, in the cube consisting of A, B, C, D, I, J, K, L, the non-rubies cannot be disconnected, so every non-ruby is connected to D, therefore I must be a ruby. Then the only square where removing B can still disconnect the rubies is $BJNF$, so N is not a ruby and F is a ruby. With the same logic for C, G is a ruby and O is not a ruby. Now the only square where removing F can disconnect a ruby is in square $BDHF$, so H must be a ruby. Since N and D must be connected by non-rubies, L and P must be non-rubies. Now we have completely fixed every square containing ruby J. However, removing J does not disconnect the rubies of any square. So this is not possible.

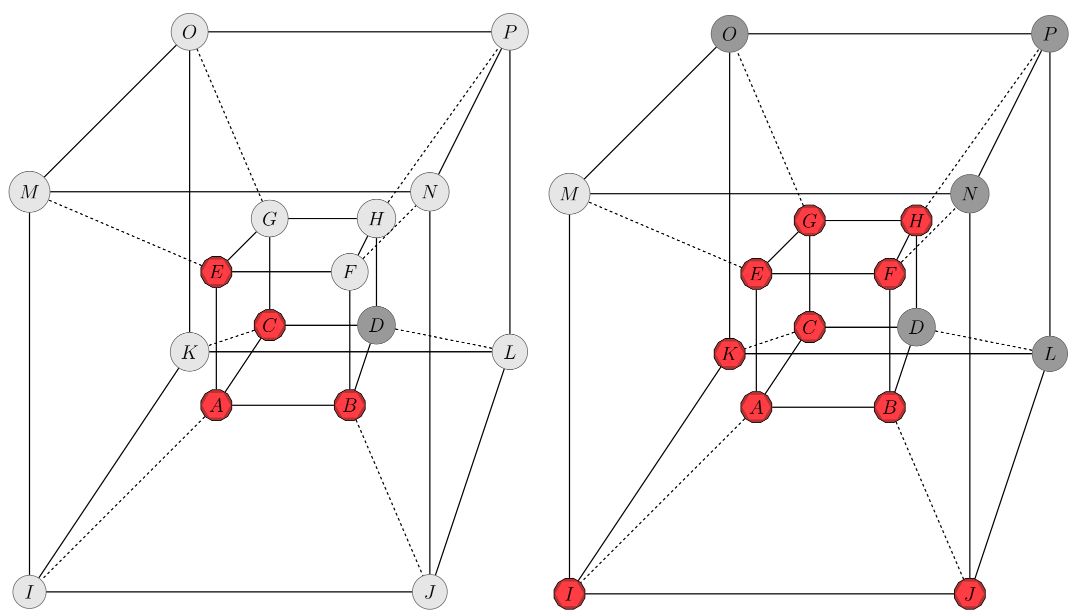


Figure 24: Red means ruby, dark grey means no ruby, and light grey is unknown. Left: A has three neighbours and one square in which it disconnects the rubies. Right: implied rubies and non-rubies we can derive from the left picture.

In conclusion, every ruby is directly connected to exactly two other rubies, so they must form a single cycle (since they must also be connected). By a cycle we mean the following: Start at a ruby and use an edge to get to one other ruby that is directly connected to it via an edge. Now, there is only one possibility to get to a ruby not visited before via an edge. This process continues until every ruby has been visited and there is one edge one can use to get from the last ruby back to the ruby at the beginning.

Now any cycle in a tesseract consists of an even number of rubies (one can see this as follows: Each time you move along an edge in one direction, you must eventually move in the opposite direction to return to the starting point). A cycle of 4 rubies is a square, in which we can remove a ruby without disconnecting them, violating rule 5. Any cycle of 6 rubies looks like the cycle $ABDHGE$, but there the non-rubies C and F are not connected in the cube $ABCDEFGH$. Any cycle of 10 or more rubies is not possible. This can be seen as follows:

Claim: Any cycle can only take a maximum of 4 rubies in a cube.

Proof. We first assume that there can be 5 or more rubies in a cube and then show that this leads to a contradiction. To illustrate the proof, we consider the cube $ABCDEFGH$. No square can contain four rubies, since each additional ruby always results in one of the rubies having three edges connected to other rubies. Therefore, due to symmetry, we can assume that the square $ABCD$ contains three rubies and the square $EFGH$ contains two or three rubies.
The rubies in the square $EFGH$ must be connected to each other through its edges. In the case of three rubies, this is obviously the case. For the case of two rubies, the connection arises due to rule 4, which dictates that the non-rubies within this cube must be connected to each other. In both cases, however, it remains impossible to place the remaining rubies such that neither the connection of the non-rubies within the cube is interrupted, nor does any ruby have three edges connecting it to other rubies. □

So the possible designs must consist of a cycle of 8 rubies. There are three types of cycles of 8 rubies: one like $ABJNPOGC$, which is the correct solution in Figure 23, one like $ABJLPOGC$ (which is not correct since L and C are not connected in square $CDLK$) and the last type has 5 rubies in the same cube ($ABJLPHGC$ or $ABJNPHGC$), but then B and H are not connected in $BDHF$. Therefore, the design is unique up to symmetries of the tesseract.

Illustration: Friederike Hofmann

14 Schlag den Staab

Author: Silas Rathke (BMS)

Challenge

Mathilda is thrilled: her favorite show, “Schlag den Staab”, is returning to television.
In the show, a contestant competes against the TV star Evan Staab in 100 games.[1] Each game is won either by the contestant or Evan Staab; a tie is not possible. The winner of the first game earns one point, the second game two points, and so on, so the winner of the final game earns 100 points. These points are added together, and whoever has the most points at the end wins the show. If the contestant and Evan Staab have the same total points at the end, a tiebreaker game is held.
Mathilda loves it when a tiebreaker happens, as it makes the show particularly exciting. After every game, the current standings are displayed, and Mathilda wonders whether it is still mathematically possible for the final score to result in a tie. Sometimes, a tiebreaker remains possible until the very last game, but more often it becomes clear much earlier that a tiebreaker is no longer possible.
Let k be the smallest positive integer such that, after the k-th game, it may no longer be possible for the final score to result in a tie. What is the last digit of k in the decimal system?

[1] In the German version, there are only 15 games, but since a polar night at the North Pole lasts six months, late-night shows are correspondingly longer...

Hint:
The Gaussian summation formula is

$$1 + 2 + 3 + \ldots + (n-1) + n = \frac{n(n+1)}{2}.$$

Possible Answers:

1. 1
2. 2
3. 3
4. 4
5. 5
6. 6
7. 7
8. 8
9. 9
10. 0

Solution

The correct answer is: 8.

We will demonstrate that $k = 68$. First, we construct cases where, for $k \geq 68$, it is **impossible** for a tie to occur, and then we show that for $k = 67$, a tie is always achievable. From this, we conclude that $k = 68$ is the solution. Because, if $k < 68$, then it would be possible that no tie can be achieved after the k-th game. This would also imply that it is impossible to achieve a tie after game 67, which contradicts the fact that we can show a tie is always possible after game 67.

1. For $k \geq 68$, it can happen that a tie is not possible:

Using the Gaussian summation formula, we know that a total of 5,050 points is available. A tie occurs if both players have 2,525 points.
We examine cases for $k \geq 68$:

- **Case $k \geq 71$:** If one player wins the first 71 games, they score 2,556 points, which already guarantees victory.
- **Case $k = 70$:** If one player wins the first 70 games, they score 2,485 points, falling short by 40 points. However, the remaining games score more than 40 points each, making a tie impossible.
- **Case $k = 69$:** If one player wins the first 69 games, they score 2,415 points, falling short by 110 points. However, after game 69 it is not possible to win to more games and score under 141 points. On the other hand, one can score a maximum of 100 points by winning only one more game. We conclude, that a tie is impossible in this scenario.
- **Case $k = 68$:** If one player wins 68 games except for game 21, they score 2,325 points, falling short by 200 points. However, starting from the 69th game, it is not possible to win three rounds and score less than 210 points. With two games, a maximum of $100 + 99 = 199$ points can be achieved, and we conclude that a tie is no longer possible in this case either.

2. For $k < 68$, there is always a possibility of a tiebreaker occurring.

We consider the score P of one of the two players after 67 games have been played and want to show that there is always a way to reach 2525 points from P. Using the Gaussian summation formula, we first obtain $P \leq \frac{67 \cdot 68}{2} = 2278$. We first explain our approach intuitively. The minimum score that can be achieved with three additional wins is $68 + 69 + 70$ or $(67 + 1) + (67 + 2) + (67 + 3)$, and the maximum is $100 + 99 + 98$ or $100 + (100 - 1) + (100 - 2)$. These compute to a minimum of 207 and a maximum of 297. Additionally, all scores in between can be achieved (this will be proven later).
Starting from 2525, a tiebreaker can be reached with three additional games if

$$2525 - 297 \leq P \leq 2525 - 207,$$

i.e., if $2228 \leq P \leq 2318$. Thus, we are left to check the cases where $P \leq 2228$.

For this, we consider the minimum and maximum number of points that can be obtained with four additional wins:

$$(67+1)+(67+2)+(67+3)+(67+4)=278,$$
$$100+(100-1)+(100-2)+(100-3)=394.$$

Again, all scores in between can be achieved (this will be proven later). This covers the cases where

$$2525-394 \leq P \leq 2525-278,$$

i.e., $2131 \leq P \leq 2247$.
We observe that the intervals for 3 and 4 additional wins overlap. Does this also hold for the intervals of 4 and 5 additional wins and, in general, for the intervals of l and $l+1$ additional wins? To answer this, we need to determine when the minimum score from $l+1$ additional wins exceeds the maximum score from l additional wins. This would allow us to cover all scores P such that their difference to 2525 falls within the range between the minimum score from 3 additional wins and the maximum score from $l+1$ additional wins. The answer is provided by the following lemma:

Lemma 1. *For $3 \leq l < 30$, the following inequality holds:*

$$(67+1)+(67+2)+\ldots+(67+(l+1)) \leq 100+(100-1)+\ldots+(100-(l-1)).$$

Proof. First, we simplify the left-hand side of the inequality:

$$\begin{aligned}(67+1)+\ldots+(67+(l+1)) &= (l+1)\cdot 67+(1+2+\ldots+(l+1))\\ &= (l+1)\cdot 67+\frac{(l+1)(l+2)}{2}\\ &= \frac{1}{2}\cdot l^2+\frac{137}{2}\cdot l+68.\end{aligned}$$

Similarly, for the right-hand side:

$$\begin{aligned}100+(100-1)+\ldots+(100-(l-1)) &= l\cdot 100-(1+2+\ldots+(l-1))\\ &= l\cdot 100-\frac{(l-1)\cdot l}{2}\\ &= -\frac{1}{2}\cdot l^2+\frac{201}{2}\cdot l.\end{aligned}$$

Thus, we are interested in the solutions to the quadratic inequality:

$$\frac{1}{2}\cdot l^2+\frac{137}{2}\cdot l+68 \leq -\frac{1}{2}\cdot l^2+\frac{201}{2}\cdot l.$$

Rearranging the inequality gives:

$$l^2-32\cdot l+68 \leq 0.$$

This corresponds to the equation of an upward-facing parabola. Letting the roots of this quadratic equation be l_1 and l_2, the inequality holds for all values between l_1 and l_2. Solving for the roots using the quadratic formula:

$$l_1 = 16-\sqrt{188} < 16-13 = 3, \quad \text{and} \quad l_2 = 16+\sqrt{188} > 16+13 = 29.$$

Thus, the statement is proven. □

By the above reasoning, all scores with a difference from 2525 that falls within the minimum score from 3 additional wins and the maximum score from 30 additional wins are covered. The maximum score from 30 additional wins is:

$$100 + (100 - 1) + \ldots + (100 - 29) = 30 \cdot 100 - \frac{29 \cdot 30}{2} = 2565.$$

Therefore, all scores P satisfying

$$2525 - 2565 \leq P \leq 2525 - 207$$

or

$$-40 \leq P \leq 2318$$

are covered. We can conclude that for all possible scores P, one of the two players can force a tie when 67 games have been played. Naturally, this also applies when fewer than 67 games have been played.

Finally, we prove that every score between the minimum and maximum score from l additional wins, after the k-th played game, can be reached:

Lemma 2. *Let k be the number of games already played, and let l be the number of additional games won. Let m be the minimum number of points achievable with l additional wins and M the maximum. Then, for every score P with $m \leq P \leq M$, there exist exactly l games with scores $s_1, s_2, \ldots, s_l$ such that*

$$s_1 + s_2 + \ldots + s_l = P.$$

Proof. If $P = m$, the statement is obviously true. Now assume $P \geq m$ but $P < M$ and that we have already proven the statement for P (we will justify this later). We now prove it for $P + 1$. If $P + 1 = M$, the statement is clearly true. Otherwise, consider scores $s_1, s_2, \ldots, s_l$ summing to P. Since we assumed the statement holds for P, such scores must exist.
Rearranging, we can assume $s_l < s_{l-1} < \ldots < s_2 < s_1$. Since $P + 1 \neq M$, there exists an index i with $1 \leq i \leq l$ such that $s_i < 100 - (i - 1)$. Otherwise,

$$s_1 + \ldots + s_l \geq 100 + \ldots + (100 - (l - 1)) \geq M.$$

Let i be the smallest index for which this holds. Then either $i = 1$, in which case $s_1 + 1, s_2, \ldots, s_l$ sums to $P + 1$. Otherwise, if $i \neq 1$, we have $s_i + 1 < s_{i-1}$ due to the assumption that i is the smallest such index. In this case, the sum of $s_1, \ldots, s_{i-1}, s_i + 1, s_{i+1}, \ldots, s_l$ equals $P + 1$, and the statement holds in this case as well.
Since we already know that the statement holds for $P = m$, we can conclude from the above argument that it also holds for $m + 1$. This, in turn, implies that the statement also holds for $m + 2$. This argument can be applied iteratively until we reach M. Thus, the statement holds for all scores between m and M. This proof method is formally known as induction. □

Illustration: Friederike Hofmann

15 Logistic Challenges

Author: Anouk Beursgens
Project: Predicting the demand for after care from hospitals

Challenge

Christmas is coming close again, so the elves are busy preparing all the presents! Due to the enormous number of presents, the elves start delivering presents three weeks in advance. That is, this year, they will be delivering during the weeks of December 2-8 (week 1), December 9-15 (week 2), and December 16-22 (week 3). The closer to Christmas, the more presents they deliver per week. However, not every day the same amount of presents is delivered. There are red and green gifts. The ratio between red and green gifts depends only on the day of the week and is the same throughout the three-week period.

To plan when each of the elves needs to work, the head elf wants to predict the number of deliveries per day. He has data on the number of presents from last year, which is given in Table 1 (given on the next page). Luckily, humans are creatures of habit, so the present delivery number for this year will abide by the following two rules:

1. The deviation of the number of presents delivered on a day of the week from the average number of presents delivered on a day during that week is the same as last year. For example, if on average three presents are delivered and on Monday one present is delivered, than the deviation for Monday from the average of that week is -2.

2. The fraction of red presents delivered on a day of the week (with respect to the total number of presents delivered on that day) will be the same as last year. For example, if on Monday there were one red and two green presents delivered, then the fraction of red presents on Monday of that week is 1/3.

	(Week 1)			(Week 2)			(Week 3)		
Day	Date	Red	Green	Date	Red	Green	Date	Red	Green
Mon	12/04	2	2	12/11	5	5	12/18	11	11
Tue	12/05	3	3	12/12	6	6	12/19	12	12
Wed	12/06	3	3	12/13	6	6	12/20	12	12
Thu	12/07	4	4	12/14	7	7	12/21	13	13
Fri	12/08	3	6	12/15	5	10	12/22	9	18
Sat	12/09	2	1	12/16	6	3	12/23	14	7
Sun	12/10	5	1	12/17	10	2	12/24	20	4

Table 1: Number of red and green presents delivered on each day during a three-week period in the previous year 2023.

For example, in Table 1 we can observe

- on each of the three weeks, Tuesday had 2 more presents delivered than Monday,
- on each of the three Mondays, half of the delivered presents were red, while on each Friday only one third was red,
- in week 1 there was a total of 42 delivered presents and in week 2 the total was 84.

One thing has changed this year: Christmas has become more popular! Thus, the head elf knows that this year the total number of presents per week triples every week during the delivery weeks. Moreover, he knows that on Monday in week 1 (December 2) the elves will need to deliver 8 red and 8 green presents. How many red and green presents are there to be delivered on Saturday in week 3 (December 21)?

Possible Answers:

1. 56 red gifts and 28 green gifts
2. 108 red gifts and 108 green gifts
3. 108 red gifts and 36 green gifts
4. 84 red gifts and 21 green gifts
5. 72 red gifts and 26 green gifts
6. 90 red gifts and 18 green gifts
7. 106 red gifts and 53 green gifts
8. 108 red gifts and 54 green gifts
9. 56 red gifts and 56 green gifts
10. 30 red gifts and 15 green gifts

Project Reference:

Predicting the demand for aftercare from hospitals is an important topic of this project. One of the approaches we use is time series analysis. In time series decomposition, we split the time series (e.g., the number of requests for aftercare per day or the number of presents delivered per day) into a trend, a seasonal component, and an error term. The trend tells us how the time series behaves over more extended periods, while a weekly seasonality shows how it fluctuates per day of the week. Besides weekly seasonality, one could have, for example, daily seasonality (e.g., in the number of patients arriving at the emergency department in a hospital per hour) or yearly seasonality (e.g., in the number of ice creams per month).

Solution

The correct answer is: 7.

From Table 1 we can conclude two useful patterns.

First, we determine the deviation of the daily number of presents over the week. In week 1, there are in total 42 presents, which means on average $42/7 = 6$ per day. If we look at Monday, we had 2+2=4 presents, which is 2 less than the daily average for that week. If we compute this for all days of week 1 using Table 1, we get the following weekly seasonality:

Monday	Tuesday	Wednesday	Thursday	Friday	Saturday	Sunday
-2	+0	+0	+2	+3	-3	0

In fact, the same deviation pattern holds for week 2 and week 3.

Second, from the last year we observe the following fraction of red presents for each day of week 1: Moreover, the same pattern holds for week 2 and week 3.

Monday	Tuesday	Wednesday	Thursday	Friday	Saturday	Sunday
1/2	1/2	1/2	1/2	1/3	2/3	5/6

It is given that on Monday of week 1 this year, we have 16 presents (8 red + 8 green). From the pattern of last year, we know that this is 2 less than the week average. Therefore, in week 1 of this year, we have, on average, 18 presents per day, and thus $7 \times 18 = 126$ presents in total. It is given that this number of presents per week triples every week. So, in week 2, we have $126 \times 3 = 378$ presents, and in week 3 a total of $378 \times 3 = 1134$ presents. The latter means that in the third week, the average is $1134/7 = 162$ presents per day. Using the daily deviations from the average, as obtained from the data from last year, we have on Saturday, 21 December, $162 - 3 = 159$ presents. Using the ratio between red and green presents, as obtained from the data from last year, we have $159 \times 2/3 = 106$ red and 53 green presents.

Illustration: Zyanya Santuario

16 The Numbers of Nazareth

Author: Matthew Maat (Unversiteit Twente)

Project: Combining algorithms for parity games and linear programming

Challenge

In the tiny village of Nazareth, which you see in Figure 25(i), there are only six streets. It is possible to walk a *cycle*, which means you walk along a number of streets, and you return to where you started (without passing a crossing twice). For example the roads a, d and f form a cycle. We denote this cycle as (adf). In total there are seven cycles, the other ones are $(bde), (cef), (abc), (adec), (abef)$ and $(bcfd)$. Note, that the starting position of a cycle does not matter, e.g. the cycles (adf) and (dfa) are regarded as the same cycle.
Now Joseph and Mary are playing a game called 'number the streets.' Each round starts with the village elder revealing a set of cycles, for example he gives them $(bde), (abc), (bcfd)$ and $(abef)$. Mary and Jpseph have a map of the village. They each write down six integers on the map, one on every street in the map.

Joseph gets one point if both of these are true:

- In each cycle that the elder gave, the largest integer that Joseph wrote is even.
- In every other cycle, the largest integer that Joseph wrote is odd.

Mary gets one point if both of these are true:

- In each cycle that the elder gave, the sum of the integers that Mary wrote is at least 0.

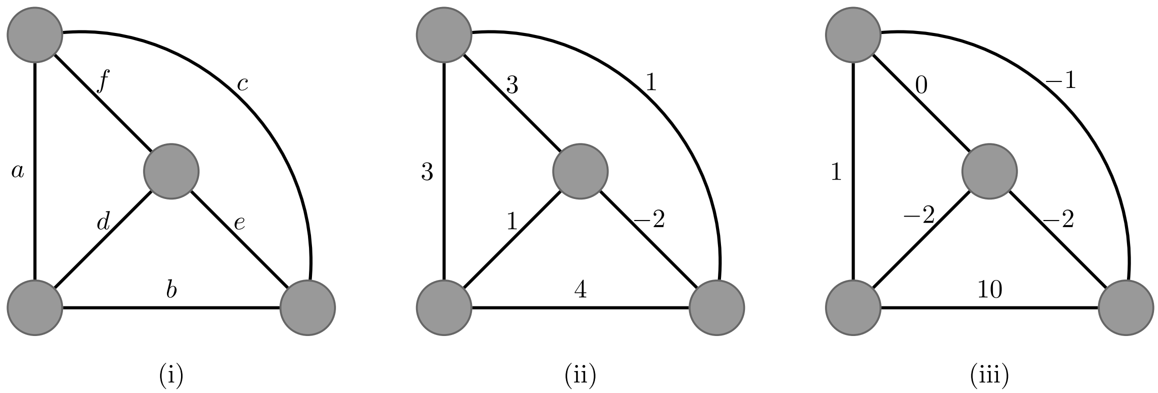


Figure 25: The streets of Nazareth

- In every other cycle, the sum of the integers that Mary wrote is less than 0.

Joseph and Mary play a practice round:
Figure 25 shows the numbers written by Joseph (ii) and Mary (iii) to score a point in the round where the eldest would have announced the roads $(bde), (abc), (bcfd)$, and $(abef)$. In the cycle (bde), which is part of the set, the highest number is 4, which is indeed even, and one can check that the two properties also hold for the other six cycles. For Maria, the sum of the numbers in cycle (bde) is $10 - 2 - 2 = 6$, which is even bigger than 0. And again, one can check that the other six cycles are also correct.
In the actual game, three rounds are played. After each round, Joseph and Mary cross out their old numbers and think of new numbers for the next round. The points scored are added up. The elder gives the following sets of cycles:

Round 1: (adf) and (cef)

Round 2: $(abc), (bde), (cef)$ and (adf)

Round 3: $(abef)$ and $(bcfd)$

Assume Joseph and Mary play as optimally as possible. Let J be the total score of Joseph and M that of Mary. What is the final score, written as $J - M$?

Possible Answers:

1. $0-3$
2. $1-3$
3. $1-1$
4. $1-2$
5. $3-3$
6. $2-1$
7. $2-2$
8. $2-3$
9. $3-0$
10. $3-1$

Project Reference:

Joseph tries to construct something close to a parity game whose winning cycles coincide with the cycles given by the elder. Mary constructs something close to a mean payoff game with the prescribed winning cycles. There are interesting connections between the two types of games.

Solution

The correct answer is: 4.

First, we provide examples of the rounds in which Mary or Joseph can score a point. Subsequently, we show that it is not possible for either of them to score any additional points.

In round 1, Joseph scores a point with the numbers in Figure 26(i), and Mary scores a point with the solution in Figure 26(ii). In round 3, Mary scores a second point with Figure 26(iii). No other points can be scored.

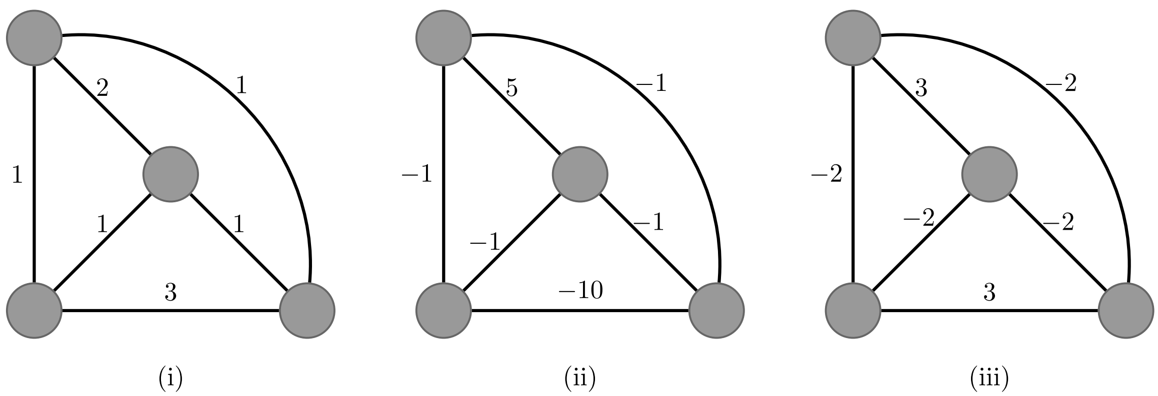


Figure 26: Possible solutions for round 1 and 3.

First of all, in round 2 and 3, every road is contained both in a cycle *in* the set, and in a cycle *not in* the set. It is therefore impossible for Joseph to score a point: If Joseph would have scored a point, what could the largest of all of his numbers be? That number will be the largest in any cycle containing it, so it should be even to make the largest number of one cycle even; but also odd to make the largest number of another cycle odd, and that is not possible of course. So Joseph cannot score more than 1 point. Finally, we show that Mary cannot score a point in round 2 by contradiction. The set of the elder consists of exactly the cycles with 3 roads. Since the sum of the roads on these cycles is at least 0, the same holds if we sum all these four cycles together:

$$(a+b+c)+(b+d+e)+(c+e+f)+(a+d+f) \geq 0$$

All the cycles with 4 roads are not in the set, so their sums each are less than 0, so the same holds for the sum of the sums:

$$(a+b+e+f)+(b+c+f+d)+(a+d+e+c) < 0$$

Note, that the term on the left side of both inequalities is the same. Thus, we get a chain of inequalities

$$0 \leq 2a+2b+2c+2d+2e+2f < 0,$$

which is impossible. So Mary cannot score a point.

Remark: Poor Joseph can never win this game duo to the connection between parity games and mean payoff games. If Joseph manages to score a point in a round with some arrangement of numbers, Mary can just write $(-5)^i$ wherever Joseph wrote i, and then her arrangement of numbers also scores a point.

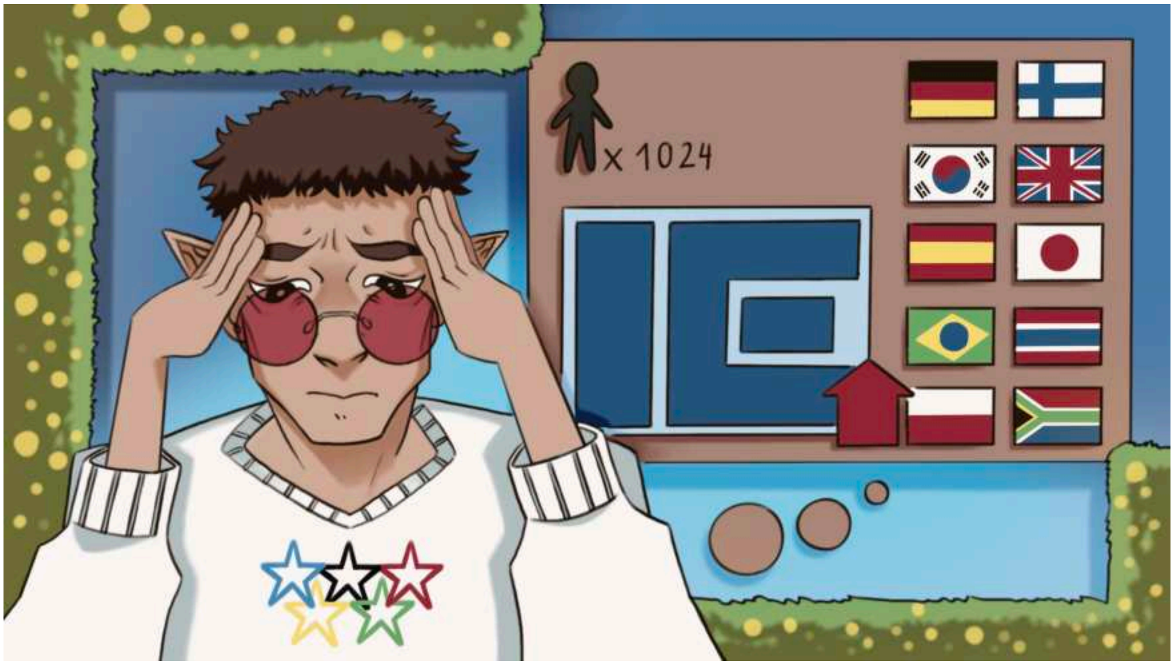


Illustration: Julia Nurit Schönnagel

The Olympic Housing Dilemma

Authors: Silas Rathke, Yamaan Attwa

Challenge

This year, the Olympic Arctic Games are held at the North Pole, attracting 1,024 athletes from across the Frozen North to compete in various sports.
Otto, the event's accommodation organizer, is tasked with housing the athletes. Therefore, he needs to follow one strict rule of the Arctic Olympic Committee: If two (distinct) athletes do not share a common language, they cannot be accommodated together in the same house. To comply with this rule, Otto asks each athlete in advance which languages they could speak.

Otto makes some interesting observations:

- There are exactly 10 languages (L_1 to L_{10}) that are spoken.
- Every possible combination of these 10 languages is spoken by exactly one athlete. This means, for example, that there is exactly one athlete who speaks only the language L_1, and similarly, there is exactly one athlete who speaks only the languages L_2, L_3, and L_8, and so on.
- Thus, there is also exactly one athlete who speaks all 10 languages, and exactly one athlete who speaks no language at all.

Of course, Otto must also work as efficiently as possible. He wants to build as few houses as possible while still adhering to the committee's rules. How many houses does Otto need to build at a minimum to accommodate all 1,024 athletes? Let k represent this minimum number of houses. What is the last digit of k in the decimal system?

Possible Answers:

1. 1
2. 2
3. 3
4. 4
5. 5
6. 6
7. 7
8. 8
9. 9
10. 0

Project Reference:

This puzzle is an example of a coloring problem and while these are particularly fun, they are usually difficult. Usually we have a certain set V whose elements we would like to color; however, the devil is allowed to give us an arbitrary list of conditions E detailing pairs that are not allowed to be colored the same. We are usually interested in determining the least possible number of colors sufficient to paint the elements of our sets without violating any of the devil's conditions. This pursuit is proved to be notoriously challenging; in fact, there is no known fast way to determine whether an arbitrary set V with a corresponding list E of conditions can be properly colored with only 3 colors. If you find a fast algorithm for that purpose, you should hire a secretary as many people would love to talk with you.

This particular problem has a pretty famous cousin: Suppose there are n languages at the north pole where every athlete speaks exactly k languages and every subset of k languages is spoken by exactly one athlete. One can now ask again for the smallest number of houses needed to accommodate the athletes so that no two with no language in common share the same house. This problem was open for 22 years before it was solved using seemingly unrelated theorems from topology!

Solution

The correct answer is: 1.

We prove that $k = 11$.

First, we show that k must be at least 11. For this, we just have to note that the one person who speaks no language and the 10 athletes who speak just 1 language all need to be in different houses. This is, because out of these 11 athletes, no pair can share a house. Therefore, just for these 11 people, Otto already has to build 11 houses.

Finally, we show that 11 houses are enough. Label the 10 languages by $L_1, L_2, \ldots, L_{10}$. For each athlete, we do the following: If he speaks L_1, we put him in house 1. If not, he proceeds to house 2. If he speaks L_2, we put him in house 2, and otherwise we continue this process. When an athlete gets to house 11, namely the one who does not speak any language, we put him in it immediately.

For houses numbered from 1 to 10, each athlete accommodated in that house speaks the corresponding language – so that this language is in common for all of them, and the rule is satisfied in houses 1–10. The only way for an athlete to not be put into one of the houses 1–10 is if he doesn't speak any of the ten languages. There is only one athlete like that, and he is situated by himself in house 11, which also satisfies the rule.

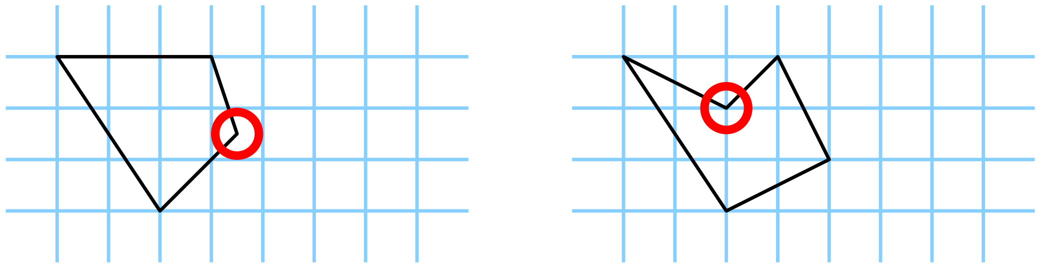
Illustration: Ivana Martić

18 Christmas Cards

Author: Christian Haase
Project: AA3-12

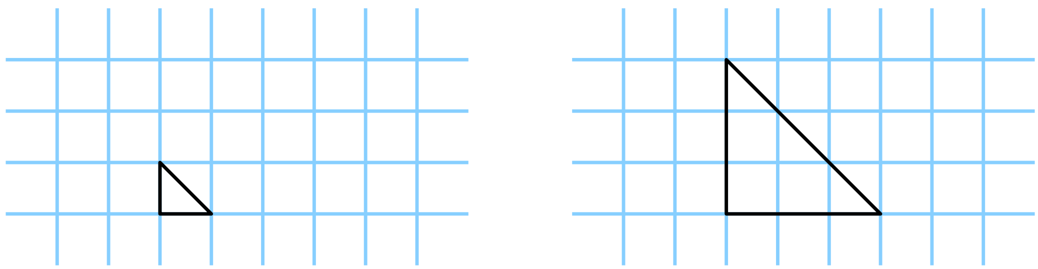
Challenge

The elves are making gift cards in various shapes for the children using graph paper, which they typically use for calculations. To make production easier, Santa has required that all cards take the form of a closed convex polygon, and all vertices of that polygon are graph-paper crossing points. Here, “convex” means that you can walk along the path making only right turns, and “vertices” means points where the elf has to change directions while cutting the paper, which restricts certain shapes from being used:
The first picture in 27 shows a convex card where a vertex is not a graph-paper crossing point, and the second picture shows an example of a non-convex card. Santa denotes by b the number of graph-paper crossing points along the boundary of the card, and by i the number of graph-paper crossing points inside the card.
Santa also wants to conserve paper, so he requires each card to have a specific area, measured in square centimeters, with each square on the graph paper having a side length of 1 centimeter. The enclosed area of each polygon is denoted by a.
Now, Santa Claus introduces the shorthand notation (b, i, a) for the cards. For instance, a gift card with 3 boundary graph-paper crossing points, zero interior graph-paper crossing points, and an area of 0.5 would have the shorthand notation $\left(3, 0, \frac{1}{2}\right)$, as shown in Figure 28 (a). Similarly, a gift card with values $(9, 1, 4.5)$ can be observed in Figure 28 (b).



(a) (b)

Figure 27: Two examples of forbidden cards: The first one has no proper vertices, and the second one is not convex. *Both maps are forbidden according to the definition!*



(a) (b)

Figure 28: Examples of valid cards

The elves are now worried that some gift cards may no longer be possible to create. Which of the following triples (b, i, a) can they still create?

- $(24, 16, 27)$
- $(15, 7, 13.5)$
- $(3, 9, 9.5)$
- $(18, 0, 9)$

Possible Answers:

1. (yes, yes, yes, yes)
2. (yes, yes, no, yes)
3. (yes, yes, yes, no)
4. (yes, no, yes, yes)
5. (no, yes, yes, yes)
6. (no, yes, no, yes)
7. (yes, no, yes, no)
8. (no, no, yes, yes)
9. (no, no, no, yes)
10. (no, no, no, no)

Project Reference:

The project aims to deepen our understanding of the expressivity of neural networks (NNs). Specifically, it establishes both lower and upper bounds on the topological simplification that a ReLU neural network can achieve given a particular architecture. In the context of general representations of functions, we observe that the function computed by a ReLU NN is piecewise linear and continuous (CPWL) since it is a composition of affine transformations and the ReLU function. Conversely, it is known that any CPWL function can be represented by a ReLU NN of logarithmic depth. In [1] it has been conjectured that this logarithmic bound is tight. In other words, there may exist CPWL functions that can only be represented by ReLU NNs with logarithmic depth. This conjecture has been proven under a natural assumption for dimension n=4 using techniques from mixed-integer optimization. Additionally, in [2] it has been shown that logarithmic depth is necessary to compute the maximum of n numbers when only integer weights are allowed. This result is based on the duality between neural networks and Newton polytopes through tropical geometry. One of the primary goals of this project is to either prove or disprove the conjecture in its full generality.

References

[1] Christoph Hertrich, Amitabh Basu, Marco Di Summa, and Martin Skutella. Towards lower bounds on the depth of ReLU neural networks. *SIAM Journal on Discrete Mathematics*, 37(2):997–1029, 2023.

[2] Christian Haase, Christoph Hertrich, and Georg Loho. Lower bounds on the depth of integral ReLU neural networks via lattice polytopes, 2023.

Solution

The correct answer is: 3.

Notice that the graph-paper lines aren't actually important– all that the elves should think about are the actual graph-paper crossings, so from now on, only the crossing points will be drawn as big dots.
By trying to draw the examples, one could notice that there exist gift cards with the wanted numbers of boundary points, interior points, and area for the first three examples, shown in Figure 29.

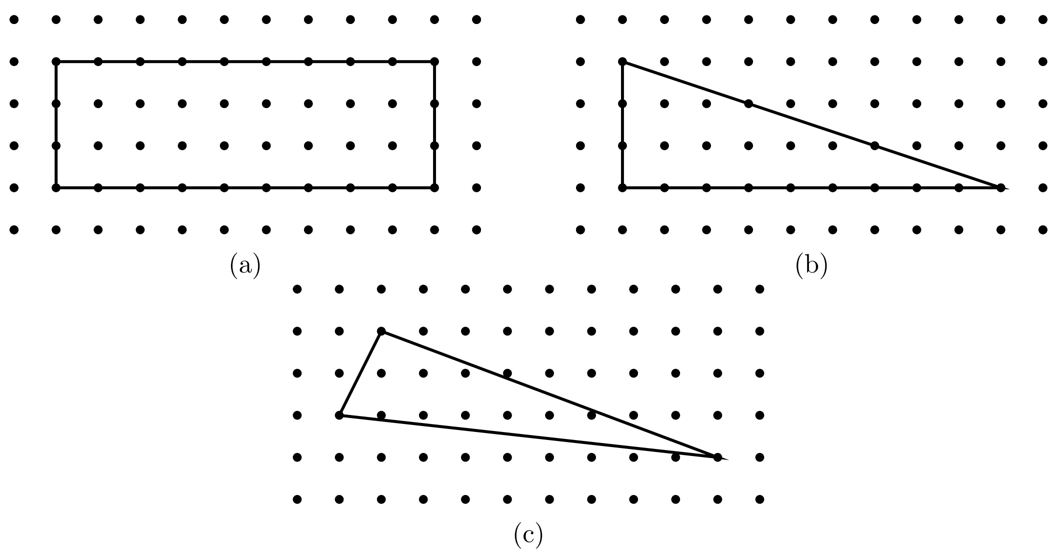


Figure 29: Examples for the first three gift cards

However hard the elves try, they can't make a gift card for the fourth example!
This follows from the theorem of Pick, which tells us that for any gift card, the following has to hold:

$$a = i + \frac{1}{2}b - 1.$$

This equation doesn't hold for the fourth example, therefore, the elves can't make a gift card of that area with those numbers of boundary and interior points.

Let's prove this theorem step by step, looking at specific possible shapes of gift cards, that can (conveniently enough) be seen from the three examples drawn in Figure 29.

(A) Rectangles with sides parallel to coordinate axes

Let's look at the rectangle with sides parallel to coordinate axes and side lengths m and n. Its area is $a = mn$. Since two of its edges have $m + 1$ and two $n + 1$ points, and only the 4 vertices of the rectangle are counted twice, the number of boundary points is $b = 2(m+1) + 2(n+1) - 4 = 2m + 2n$. In the inside of this rectangle, there are $n - 1$ rows of $m - 1$ points, therefore the number of interior points is $i = (m - 1)(n - 1) = mn - m - n + 1$. Now, we want to check if the formula $a = i + \frac{1}{2}b - 1$ holds:

$$a = mn = (m - 1)(n - 1) + m + n - 1$$

$$= (m-1)(n-1) + \frac{1}{2}(2m+2n) - 1 = i + \frac{1}{2}b - 1$$

Therefore, for all gift cards that have the shape of a rectangle with sides parallel to the coordinate axes, this formula holds.

(B) Triangle with two sides parallel to coordinate axes

Notice that every triangle like this is actually half of a rectangle with sides parallel to coordinate axes, like shown in Figure 30.

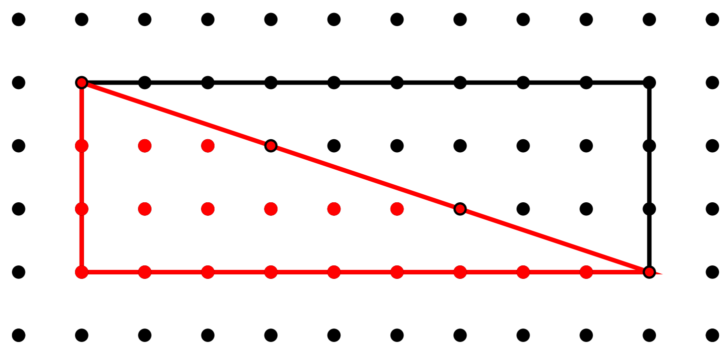


Figure 30: Gift card in the shape of a triangle with two sides parallel to coordinate axes

Let's denote by $a(T), b(T)$ and $i(T)$ the area, boundary points and interior points of the triangle, and by $a(R), b(R)$ and $i(R)$ those of the rectangle. Since we know that the formula is correct for the rectangle, we want to express $a(T), b(T)$ and $i(T)$ using $a(R), b(R)$ and $i(R)$. Since the red and black triangles in the picture are equivalent, they have the same numbers of interior and boundary points, as well as the same area, i. e. $a(T) = \frac{1}{2}a(R)$. Denote by e the number of points on the common edge of the two triangles, which is also the diagonal of the rectangle. All of those points are counted as boundary points of both of the triangles, but in the rectangle, only the two endpoints of this line are in the boundary, the other points are in the interior. Summing up the boundary points of the two triangles, we get all of the boundary points of the rectangle, where we counted the two endpoints of the diagonal twice, as well as additionally adding $e-2$ interior points of the rectangle two times. Therefore,

$$2b(T) = b(R) + 2 + 2(e-2) = b(R) + 2e - 2. \tag{4}$$

Dividing by two, we get $b(T) = \frac{1}{2}b(R) + e - 1$. The interior points of the rectangle are exactly the interior points of the two triangles plus all the points on the diagonal excluding the endpoints, i. e.

$$i(R) = 2i(T) + e - 2. \tag{5}$$

Rearranging and dividing by two, we get $i(T) = \frac{1}{2}i(R) - \frac{1}{2}e + 1$.
Next, we combine equations (4) and (5) to calculate $i(T) + \frac{1}{2}b(T) - 1$:

$$i(T) + \frac{1}{2}b(T) - 1 = \frac{1}{2}i(R) - \frac{1}{2}e + 1 + \frac{1}{4}b(R) + \frac{1}{2}e - \frac{1}{2} - 1 =$$

$$= \frac{1}{2}i(R) + \frac{1}{4}b(R) - \frac{1}{2} = \frac{1}{2}(i(R) + \frac{1}{2}b(R) - 1) = \frac{1}{2}a(R),$$

where for the last equality we use Pick's formula for rectangles, derived before in (A). It follows $i(T) + \frac{1}{2}b(T) - 1 = a(T)$, so the formula also holds for all triangles with two sides parallel to coordinate axes.

(C) Gluing two giftcards together on an edge

In the previous, we didn't really use the fact that we are working with triangles and rectangles that much – we mostly used the fact that to get the rectangle, we glued the two triangles together on an edge that they share.
When two gift cards are glued together along a common edge, the resulting area is equal to the sum of the areas of the two individual cards. Similarly, since the equation $a = i + \frac{1}{2}b - 1$ holds for each card, we observe that the left-hand side of their respective equations combines linearly. Our goal is now to demonstrate that the right-hand side of the formula also combines linearly under this operation.
Let P and Q be gift cards that we are gluing together on a common edge, and call the gift card that we get S, and let e be the number of points on the common edge. As mentioned, we know $a(S) = a(P) + a(Q)$.
All of the points on the common edge are boundary points of both triangles P and Q, but in S, only the two endpoints of the diagonal are in the boundary and the remaining points are in the interior. Summing up the boundary points of P and Q, we get the boundary points of S, with the two endpoints of the common edge counted twice, and the additional $2(e-2)$ points that are in the interior of S. Therefore, $b(P) + b(Q) = b(S) + 2 + 2(e-2)$, and by rearranging we get $b(S) = b(P) + b(Q) - 2e + 2$.
The interior points of S are exactly the interior points of P plus the interior points of Q, plus the $e-2$ points on the common edge that aren't endpoints, i. e. $i(S) = i(P) + i(Q) + e - 2$. Putting all of this together, we get:

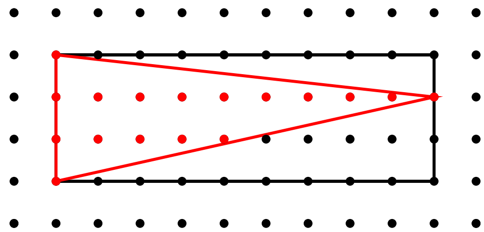
$$i(S) + \frac{1}{2}b(S) - 1 = i(P) + i(Q) + e - 2 + \frac{1}{2}(b(P) + b(Q) - 2e + 2) - 1,$$

and finally, by rearranging we obtain:

$$i(S) + \frac{1}{2}b(S) - 1 = (i(P) + \frac{1}{2}b(P) - 1) + (i(Q) + \frac{1}{2}b(Q) - 1).$$

Arbitrary triangles

We want to show that the formula also holds for the gift cards in shapes of triangles that have one or none of their edges parallel to the coordinate axes. The idea is, around every triangular gift card, to prescribe a rectangle with sides parallel to the coordinate axes by gluing some triangles with two sides parallel to the coordinate axes to the starting triangle. Specifically, this rectangle will be bounded exactly by the lines $\{x = \text{maximal } x \text{ coordinate of the vertices of the triangle}\}$, $\{x = \text{minimal } x \text{ coordinate of the vertices of the triangle}\}$, $\{y = \text{maximal } y \text{ coordinate of the vertices of the triangle}\}$ and $\{y = \text{minimal } y \text{ coordinate of the vertices of the triangle}\}$.
Since both a and $i + \frac{1}{2}b - 1$ combine linearly when gift cards are glued along common edges, and the formula holds for rectangles with edges parallel to the coordinate axes as well as for triangles with two edges parallel to the coordinate axes, it follows that the formula


(a)

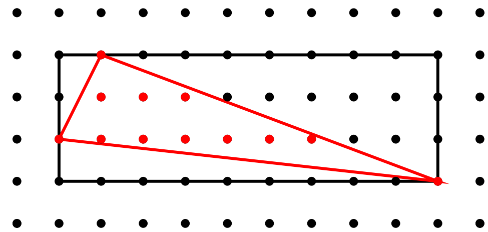

(b)

Figure 31: Rectangular gift cards prescribed around arbitrary triangular gift cards

Arbitrary convex polygon

Any gift card that is of a shape of a convex polygon can be split into a gluing of triangular gift cards by drawing all diagonals that start from the same vertex.

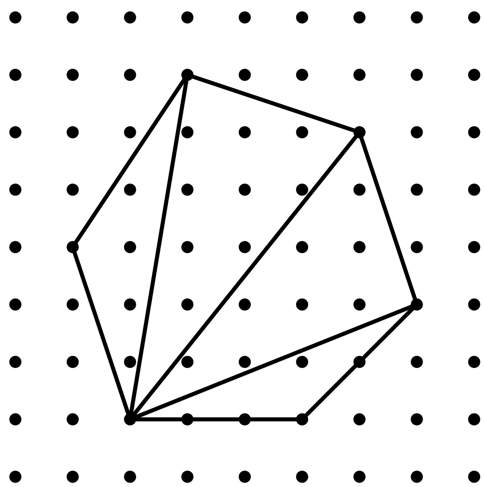


Figure 32: Gift card split into triangular gift cards

Therefore, since we proved that the formula holds for all triangle gift cards, and that if we glue together gift cards for which the formula holds, it also holds for the new gift card, we can conclude that the formula will hold for all of the gift cards.

Illustration: Zyanya Santuario

19 An Intertwined Issue

Author: Lukas Protz (TU Berlin, MATH+)
Project: MATH+

Challenge

It is dark and dusty. What else does one expect from Santa's attic at the North pole? Only equipped with a small flashlight, the eager elf Eifi wanted to clean the attic of Santa's workshop so that there was more room for all the presents to be stored. To his surprise, the attic is full of little treasure chests sealed by a weird mechanism. Eifi is wondering whether Santa is hiding something from the elves, and eager as he is, he tries to open the chests. He discovers that the mechanism works as follows:

- On the lid of each chest are three cogs. The number of teeth on each cog can be equal or different from the numbers of teeth of the other two cogs.
- When looking directly down at the cogs, one tooth of each gear always points straight up.
- Additionally, there are three buttons on the box. When a button is pressed, two of the cogs rotate simultaneously. The rotation always occurs clockwise, moving each cog forward by one tooth. Buttons cannot be pressed simultaneously but only one after another.
- The cogs are assigned to the buttons as follows:
  - Button 1 rotates cogs 2 and 3

  - Button 2 rotates cogs 1 and 3
  - Button 3 rotates cogs 1 and 2
- The box opens when all three marked teeth point upwards.

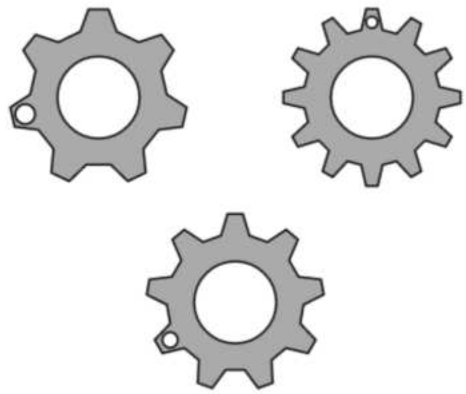


Figure 33: Example of a starting configuration of three cogs. Only the marked tooth of the upper right cog is facing upwards.

Unfortunately, Eifi has no strategy when trying to unlock the chests and only manages to open a few chests. But that does not stop Eifi from continuing his quest to uncover all the secrets that Santa may hide in them. If only there were a universal manual on how to open all the chests...

Can you help Eifi? For which combinations of three cogs with a number of teeth a, b, and c is it possible to open the chest, independent of the starting position of each cog?

Remark: The question asks for the largest set of triples a, b and c or equivalently for the smallest sufficient restriction on a, b and c.

Possible Answers:

1. It is never possible.
2. a, b and c are pairwise coprime (share no common factor).
3. Two of the numbers a, b and c are coprime.
4. a, b and c are not divisible by 2.
5. Not all of the numbers a, b and c are divisible by 2.
6. a, b and c are all divisible by 2.
7. a, b and c are not divisible by 3.
8. Not all of the numbers a, b and c are divisible by 3.
9. a, b and c are all divisible by 3.
10. None of the other possible answers apply.

Solution

The correct answer is: 5.

First we show, that all triples (a, b, c) which do not satisfy statement 5 cannot always be opened. In other words, we show that if all numbers a, b, c are even, then one cannot always open the chest.
To show this, we use the following notation:

- $(0, 0, 0)$ refers to the state, where all marked teeth point upwards.
- (x, y, z) refers to the state obtained from $(0, 0, 0)$ by rotating the first cog x times, the second cog y times and the third cog z times. The order of the cogs can be arbitrary, but once chosen they will not be changed anymore.
- $(x, y, z) \rightsquigarrow (x', y', z')$ refers to the possibility of rotating the cogs with the buttons such that state (x, y, z) can be transformed into state (x', y', z'). Else we write $(x, y, z) \not\rightsquigarrow (x', y', z')$.

Claim 1: $(1, 0, 0) \not\rightsquigarrow (0, 0, 0)$, if a, b, c are all divisible by 2.

Proof: The sum of the starting positions p and all rotations r (1 button press "=" 2 rotations) has to result in a sum s of complete rotations of the cogs (Otherwise the special teeth do not point upwards). In our case $p = 1$, which is odd. Moreover, r is always even. Thus, $r + p$ is odd. On the other hand, each complete rotation of a cog corresponds to an even number, as the number of teeth on each cog is divisible by 2. Thus, an arbitrary sum of complete rotations of the cogs is even. But then the odd number $r + p$ equals an even number s, which is a contradiction. Therefore, $(1, 0, 0) \not\rightsquigarrow (0, 0, 0)$. □

Next, we show that if statement 5 holds, i.e. one of a, b, c is not divisible by 2, then one can always open the chest. To do this, we assume that the number of teeth of the first cog is not divisible by two (otherwise we choose a different order of the cogs). One simple observation is that it is always possible to transform a state (x, y, z) to a state $(x', 0, 0)$ with x' being dependent on x, y and z. This transformation can be done by pressing the button for the first and second cog multiple times, until the second cog makes a complete rotation. Similarly, one presses the button for the first and third cog, until the third cog performs a complete rotation.

The last thing to show is the following claim:

Claim 2: $(x, 0, 0) \rightsquigarrow (0, 0, 0)$, if a is not divisible by 2.

Proof:
Step 1: Press the buttons for the first and second cog and for the first and third cog alternately until one of the two following cases occurs:

- Case 1: The first cog finishes its next complete rotation and both buttons were pressed equally often. In this case proceed with step 2.
- Case 2: The first cog finishes its next complete rotation and one button was pressed once more than the other. In this case, continue pressing the buttons alternately until

the first cog again completes a full rotation and then proceed with step 2. Because the number of teeth of the first cog is odd, the next time the first cog finishes a complete rotation, both buttons were pressed equally often.

In both cases, one can achieve that the two buttons involved were pressed the same number of times, and the marked tooth on the first cog is pointing upwards.

Step 2: Press the remaining button until both the second and the third cog make a full number of complete rotations. This is really possible as can be seen in the following way: After step one, each of the second and third cog were rotated equally often. Let us call this number n. Now, the second cog will finish a complete number of rotations after $b-n,\ 2b-n,\ 3b-n,\dots$ presses of the remaining button (provided that these quantities are non-negative). Similarly, the third cog finishes a complete number of rotations after $c-n,\ 2c-n,\ 3c-n,\dots$ presses of the remaining button. In both sequences, the numbers $b\cdot c-n,\ 2\cdot b\cdot c-n,\ 3\cdot b\cdot c-n,\dots$ appear. Thus, choosing the first positive number in the last sequence results in both, the second and third cog, finishing a complete number of rotations. Hence, all marked teeth point upwards and the chest opens. This completes the proof of the claim. □

Let us hope that Eifi only encounters chests where at least one cog has an odd number of teeth and let's hope that someone shows him how to open the chests systematically...

Bonus: The question can be generalized to an arbitrary number n of cogs on each chest and each button rotating k cogs simultaneously. What restrictions are there on the number of teeth for the cogs?

Illustration: Julia Nurit Schönnagel

20 Christmas Market Visit

Author: Sören Nagel (ZIB)
Project: EF 45-1

Challenge

It is the night before Christmas, and Santa's elves are busily working on producing gifts. While they work, they discuss which Christmas market they should visit together after their shift. The decision is not easy:

- **Christmas Market M** is famous for its roasted almonds.
- **Christmas Market A** is known for its candied apples.

Nine elves are seated in a row, and each has a preference for one of the markets. However, some of them have a strong, unchangeable opinion, while others are undecided and influenced by their neighbors. Here's the situation:

- **Elf Max** sits far left (on the first chair) and is a devoted fan of roasted almonds. He will never change his opinion.
- **Elf Anna** sits far right (on the last chair) and is equally steadfast in her love for candied apples.

The elves in between (elves 2 to 8) have no fixed opinions and adjust their preference based on their neighbors. Initially, their preferences are randomly assigned to either roasted almonds or candied apples.
Each round, one of the undecided elves from positions 2 to 8 is chosen randomly. This elf, in turn, randomly selects one of their two direct neighbors and adopts their opinion. The next round then begins, where another randomly selected undecided elf from positions 2 to 8 adopts the preference of one of their neighbors.
The first question is:

1. What is the probability that, after a very long discussion, the majority of elves will decide in favor of roasted almonds?

In the following year, Max wants to influence the group's decision-making process more effectively but still has no allies supporting roasted almonds. To increase his influence, he decides to move two spots over to the third position by swapping places with the undecided elf who was previously there. Although Max now has a more central position, he is still the only steadfast supporter of roasted almonds, and Anna, still sitting far right, remains convinced to visit Christmas Market A.

The second question is:

2. After Max's change of position, what is the probability that the majority of elves will decide in favor of roasted almonds after a very long discussion?

The answer is rounded to three decimal places.
Mathematically, the probability stabilizes in the sense that it no longer changes. Discussions may only stop once the probability has stabilized.

Hint: It can be assumed that the stabilized probability that an undecided elf chooses roasted almonds or candied apples depends linearly on their position between the stubborn elves Max and Anna.
For example, if there were only 4 elves and we number Max as 0 and Anna as 3, the stabilized probabilities for apples, which Anna wants, would be as follows:

- Elf 0 (Max): $\frac{0}{3} = 0$,
- Elf 1 (undecided): $\frac{1}{3}$,
- Elf 2 (undecided): $\frac{2}{3}$,
- Elf 3 (Anna): $\frac{3}{3} = 1$.

Possible Answers:

1. 0 and 0
2. 0 and 0.153
3. 0.3 and 0.666
4. 0.5 and 0.153
5. 0.5 and 0.666
6. 0.5 and 0.75
7. 0.5 and 0.85
8. 1 and 0.153
9. 1 and 0.666
10. 1 and 0.866

Project Reference:

This puzzle examines how position and influence affect consensus formation in groups with a mix of determined and undecided members. It serves as an illustrative example of the close connection between network structure and dynamics. The position of actors within an interaction network, as well as their interactions, can determine their influence, and both are essential components in the dissemination of information in social networks or the spread of innovations along ancient and modern road or trade networks.

Solution

Question 1:

Note that the entire problem is symmetric – the only steadfast elves sit at opposite ends of the row, and all other elves initially have a random opinion with equal probability. The entire discussion process favors neither roasted almonds nor candied apples. Since Max's and Anna's goals are identical – to convince at least 4 out of the 7 elves numbered 2 to 8 – their chances of success are also identical. Therefore, the probability that the group chooses roasted almonds is 0.5.

Question 2:

This scenario drastically changes the dynamics, as Max is now sitting more centrally and potentially has greater influence over the majority of the flexible elves. Let us evaluate how this affects the probability of a majority choosing roasted almonds.
If Max sits in position 3, the following occurs: At some point during the long discussion, the second elf is chosen and randomly speaks with Max, after which the first elf is consulted. The only neighbor the first elf can ask is the second elf, and therefore, the first elf also begins to favor roasted almonds. Once this happens, the elves in seats 1 to 3 all prefer roasted almonds and do not change their minds anymore.
From this point onward, Max and Anna have 5 elves between them who do not have a fixed choice. Note that the problem is no longer symmetric – for Anna to win, she must convince 4 of these elves, while Max only needs to convince 2.
We label these 5 undecided elves as 1 through 5, Max as 0, and Anna as 6. Note that this problem is linear due to equal distribution and the long discussion duration – the probability that elf number i ultimately chooses candied apples is

$$p_i = \frac{i}{6}.$$

The linearity reflects the relative proximity of an elf to Max or Anna: The closer an elf is to Anna, the higher the probability of choosing candied apples, and vice versa.
For example, Anna at position 6 has a probability of $\frac{6}{6} = 1$, as she is fully convinced of candied apples. Max at position 0 has a probability of $\frac{0}{6} = 0$, as he exclusively chooses roasted almonds. An elf exactly in the middle, at position 3, has a probability of $\frac{3}{6} = 0.5$ to choose candied apples. This shows that the probabilities along the row are proportional to the position of an elf between Max and Anna, with the values increasing linearly from 0 to 1.
We define the vector $O := (x_1, x_2, \ldots, x_9)$ as the binary vector of opinions, where $x_i = 1$ means that elf i is in favor of roasted almonds, and $x_i = 0$ means that elf i is in favor of candied apples. Consider the following stable configurations:

1. $O := (1, 1, 1, 0, 0, 0, 0, 0, 0)$
2. $O := (1, 1, 1, 1, 0, 0, 0, 0, 0)$
3. $O := (1, 1, 1, 1, 1, 0, 0, 0, 0)$
4. $O := (1, 1, 1, 1, 1, 1, 0, 0, 0)$
5. $O := (1, 1, 1, 1, 1, 1, 1, 0, 0)$

6. $O := (1,1,1,1,1,1,1,1,0)$

In 4 out of 6 cases, the majority decides in favor of the almond Christmas market, and in 2 out of 6 cases, in favor of the apple Christmas market. We also note that the opinion of an elf can only change if this elf is seated between two neighbors with differing opinions. As a result, the states only transition between these 6 configurations. Therefore, we conclude that with a probability of $\frac{4}{6} = \frac{2}{3}$, the majority decides in favor of the almond Christmas market.
In the next step, we show that all other states converge to the described states.
Consider states in which an elf with opinion A sits between two neighbors who both have the same opinion, namely B. In such a case, this elf will immediately adopt the opinion of its neighbors because this elf will eventually be called upon. In fact, the probability for this elf to never be called upon is 0.
Thus, only the following states remain:

1. $O := (1,1,1,0,0,1,0,0,0)$
2. $O := (1,1,1,0,0,1,1,0,0)$
3. $O := (1,1,1,0,0,0,1,1,0)$
4. $O := (1,1,1,0,0,1,1,1,0)$
5. $O := (1,1,1,1,0,0,1,1,0)$

In these states, there exist isolated elves who are surrounded by elves with other opinions. However, the isolated elves will almost certainly adopt the opinion of their neighbors over the course of the discussion. Once one of the isolated elves is convinced, it influences the others. Over the course of the discussion, the probability that none of the isolated elves is called upon becomes vanishingly small. Thus, every initial state converges to one of the six stable states after a sufficiently long time.
In summary, the 6 stable states are the only possible final states, with the majority deciding in favor of the almond Christmas market in $\frac{4}{6} = \frac{2}{3}$ of the cases.

Illustration: Ivana Martić

21 Wise Thanks to AI?

Authors: Moritz Grillo, Martin Skutella
Project: AA3-12: On the Expressivity of Neural Networks

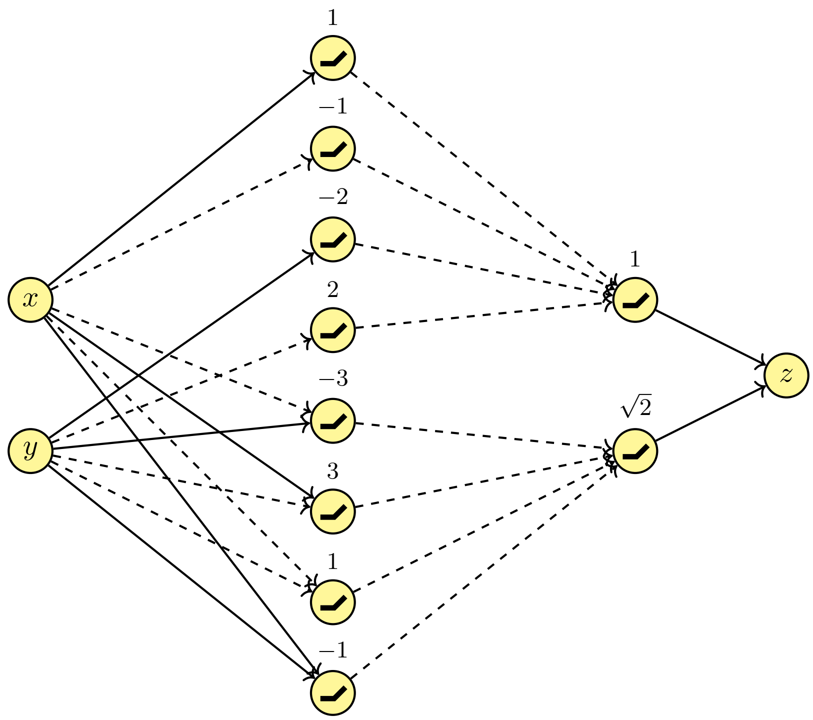
Challenge

In ancient times, there was a story of three wise men guided by a star to a specific location—a tale passed down for generations. This idea of a star leading people to an exact spot had always baffled high school senior Geoffrey. He didn't put much stock in the centuries-old tale, dismissing it as little more than an ancient "urban legend." But the logistics of the star left him curious. Even without modern light pollution, how could a faraway celestial object possibly serve as a precise guide?

Determined to modernize the concept, Geoffrey and his friend John decided to bring some tech flair to their town's holiday play. Using artificial intelligence, they designed a small neural network and packaged it as a smartphone app. The app was handed to the three actors playing the wise men, who were chosen by the local event organizer. Unfortunately, the chosen actors neither fit the "wise" nor the "reliable" description.

Chantalle, on the other hand, who had been cast as the leading role of Mary, was much more engaged. According to the organizer's directions, her role was to stay silent, looking serene while her on-stage husband, Joseph, did most of the talking. Frustrated by this outdated division of roles, Chantalle decided to channel her energy elsewhere: helping ensure the so-called 'wise men' could actually find their way to the target destination. She had her reasons, too—she was eager to wrap up the play and step away from her silent role as quickly as possible.

"It's actually pretty straightforward," John explained. "You enter the x- and y-coordinates of your location into the neural network, and it calculates a non-negative value z. The closer you are to the target spot, the larger z will be."
"And how does the neural network calculate z?" Chantalle asked with interest. "And what value of z means you've reached the target?" Geoffrey eagerly pulled out a detailed diagram to show her the inner workings of their neural network:



"The network consists of interconnected neurons," Geoffrey explained, "which are represented here as yellow circles, and their connections as arrows. The neurons labeled x and y on the left are input neurons, where the values of the x- and y-coordinates are entered. The neuron labeled z on the right is the output neuron, which outputs the value z. Between them are two layers of hidden neurons, where the actual computation takes place. The first layer consists of eight, and the second of two hidden neurons."
John continued: "To compute the values of neurons in one layer, you need the values of the neurons in the previous layer. These are passed to the next layer via the arrows. A solid arrow transmits the value of a neuron to the neuron in the next layer. A dashed arrow also transmits this value but flips the sign, i.e., multiplies it by -1. The receiving neuron adds all the values transmitted by the incoming arrows to the number written above the neuron. The value of the receiving neuron is then the maximum of 0 and this sum. For example, if x and y are the values of the input neurons, the values of the eight hidden neurons in the first layer are (from top to bottom):

$$\max\{0, x+1\}$$
$$\max\{0, -x-1\}$$
$$\max\{0, y-2\}$$
$$\max\{0, -y+2\}$$
$$\max\{0, -x+y-3\}$$
$$\max\{0, x-y+3\}$$
$$\max\{0, -x-y+1\}$$
$$\max\{0, x+y-1\}$$

The values of the two hidden neurons in the second layer are calculated similarly from the values of the hidden neurons in the first layer. The value z output by the output neuron is the sum of the values of the two hidden neurons in the second layer."
Fascinated, Chantalle studied the sketch, trying to grasp the intuitive meaning of the network's neurons. Geoffrey suggested she figure out for which coordinate pairs x, y the two hidden neurons in the second layer take a strictly positive value. "From that, you can immediately determine for which pairs x, y the resulting output z is strictly positive," he said. When Chantalle eventually drew a sketch on her notepad, Geoffrey recognized it and knew she had understood.
"And now I also know at which coordinate pair (x, y) the three kings will find the manger, and what the maximum value for z is there," said Chantalle, her devout expression replaced by a satisfied grin.

Task

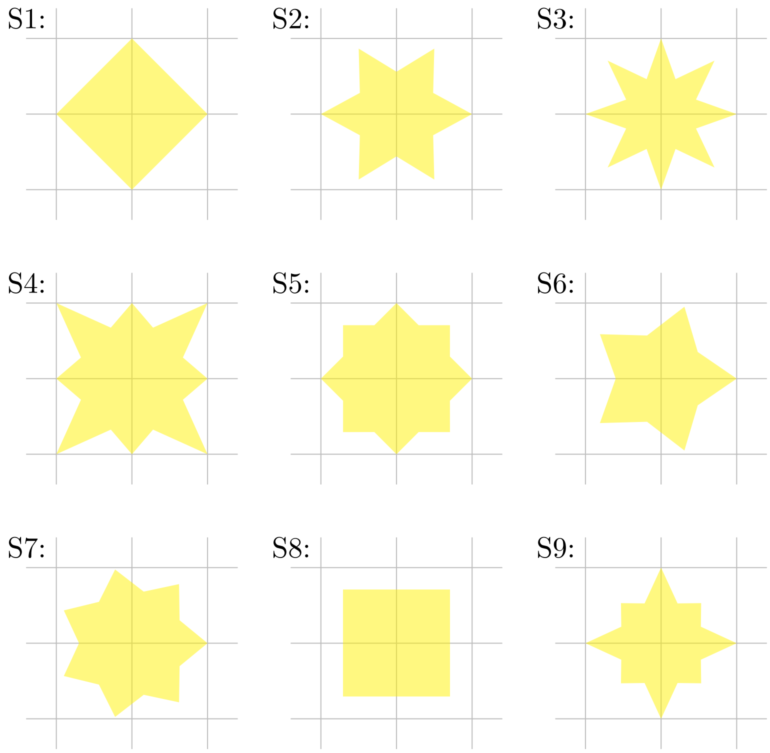
Determine

(i) which of the sketches S1 through S9 in Figure 34 a) below is correct,

(ii) which of the values W1 through W9 in Figure 34 b) below is the correct maximum value for z, and

(iii) which coordinate pair P1 through P9 in Figure 34 c) below corresponds to this maximum value.

a) Possible sketches of the region in the x-y plane where $z > 0$
(Note, that the sketches are to scale):



b) Possible maximum values for z:

W1: 0	W2: 1	W3: 2
W4: 3	W5: 4	W6: $\sqrt{2}$
W7: $\sqrt{2}+1$	W8: $\sqrt{2}+2$	W9: $2\sqrt{2}$

c) Possible coordinate pairs (x, y) where the maximum value is attained:

P1: $(0, 0)$	P2: $(0, 1)$	P3: $(1, 2)$
P4: $(2, 1)$	P5: $(-1, 2)$	P6: $(-2, 1)$
P7: $(1, -2)$	P8: $(2, -1)$	P9: $(-1, -2)$

Figure 34: Possible sketches, values, and coordinate pairs.

In which of the answer options 1 to 10 does the correct triplet consisting of sketch, value, and coordinate pair appear?

Possible Answers:

1. (S1,W1,P1) or (S2,W2,P2) or (S3,W3,P3) or (S4,W4,P4)
2. (S5,W5,P5) or (S6,W6,P6) or (S7,W7,P7) or (S8,W8,P8)
3. (S9,W9,P9) or (S1,W3,P1) or (S2,W4,P2) or (S3,W5,P3)
4. (S4,W6,P4) or (S5,W7,P5) or (S6,W8,P6) or (S7,W9,P7)
5. (S8,W1,P8) or (S9,W2,P9) or (S1,W2,P3) or (S2,W3,P3)
6. (S3,W4,P5) or (S4,W5,P6) or (S5,W6,P7) or (S6,W7,P8)
7. (S7,W8,P9) or (S8,W9,P1) or (S9,W1,P2) or (S1,W5,P4)
8. (S2,W7,P5) or (S3,W6,P6) or (S4,W8,P7) or (S5,W9,P8)
9. (S6,W1,P9) or (S7,W2,P1) or (S8,W3,P2) or (S9,W4,P3)
10. (S1,W9,P5) or (S2,W8,P6) or (S3,W7,P5) or (S4,W6,P8)

Project Reference:

In our MATH$^+$ research project titled *On the Expressivity of Neural Networks*, we are working on a better structural understanding of neural networks. Similar to Chantalle in our story, we aim to understand the properties of a given neural network. Furthermore, we explore which structures a neural network should have in order to compute certain functions exactly. One of our results, for example, states that neural networks with many hidden layers can have significantly more complex decision patterns (the regions where the neural network takes on a positive value) than neural networks with fewer hidden layers.
A few weeks ago, Geoffrey Hinton from the University of Toronto (Canada) and John Hopfield from Princeton University (USA) were awarded the Nobel Prize "*for fundamental discoveries and inventions that enable machine learning with artificial neural networks*".

Solution

The correct answer is: 4, as the correct triplet is (S5,W7,P5).

Chantalle first calculates the value z_1, which the upper hidden neuron in the second layer assumes as a function of the input values x and y. To do this, she sums the values of the first four neurons from the first layer (given at the top of the task), multiplies this by -1, adds 1, and computes the maximum of the result and 0.
She takes into account the following properties:

$$\max\{0, x+1\} + \max\{0, -x-1\} = \max\{x+1, -x-1\} = |x+1|,$$

and

$$\max\{0, y-2\} - \max\{0, -y+2\} = \max\{y-2, -y+2\} = |y-2|.$$

Thus, the sum of the four neurons becomes $|x+1| + |y-2|$, and therefore,

$$z_1 = \max\{0, -|x+1| - |y-2| + 1\}.$$

For the value z_2, which is assumed by the lower neuron in the second layer, Chantalle proceeds analogously. She sums the values of neurons 5–8 from the first layer, multiplies this by -1, adds $\sqrt{2}$, and computes the maximum of the result and 0. The sum of the four neurons multiplied by -1 corresponds to $-(|x-y+3| + |x+y-1|)$.
To simplify the expression, Chantalle uses the fact that

$$\pm(x-y+3) \pm (x+y-1) \leq |x-y+3| + |x+y-1|,$$

and that at least one of the four cases on the left leads to equality with the term on the right. She calculates the four different cases:

$$\begin{aligned}
(x-y+3) + (x+y-1) &= 2x+2, \\
(x-y+3) - (x+y-1) &= -2y+4, \\
-(x-y+3) + (x+y-1) &= 2y-4, \\
-(x-y+3) - (x+y-1) &= -2x-2.
\end{aligned}$$

Thus, it follows that

$$-(|x-y+3| + |x+y-1|) = -\max\{-2y+4, -2x-2, 2x+2, 2y-4\},$$

which becomes

$$\begin{aligned}
-(|x-y+3| + |x+y-1|) &= -\max\{-2y+4, -2x-2, 2x+2, 2y-4\} \\
&= -2 \cdot \max\{\max\{y-2, -y+2\}, \max\{x+1, -x-1\}\} \\
&= -2 \cdot \max\{|y-2|, |x+1|\}
\end{aligned}$$

Therefore,

$$z_2 = \max\{0, -2 \cdot \max\{|y-2|, |x+1|\} + \sqrt{2}\}.$$

Since the absolute value of a number can never be negative, it follows that

$$z_1 = \max\{0, -|x+1| - |y-2| + 1\} \leq \max\{0, 1\} = 1,$$

and similarly, $z_2 \leq \sqrt{2}$ for all x and y. In particular, it always holds that

$$z = z_1 + z_2 \leq 1 + \sqrt{2}.$$

Conversely, for $x = -1$ and $y = 2$, it follows that

$$|x + 1| = |y - 2| = 0,$$

and thus $z = 1 + \sqrt{2}$. For every other pair (x, y), either $|x + 1| > 0$ or $|y - 2| > 0$, implying that $z < 1 + \sqrt{2}$. **Thus, $(-1, 2)$ are the unique target coordinates, and the maximum function value is $1 + \sqrt{2}$.**

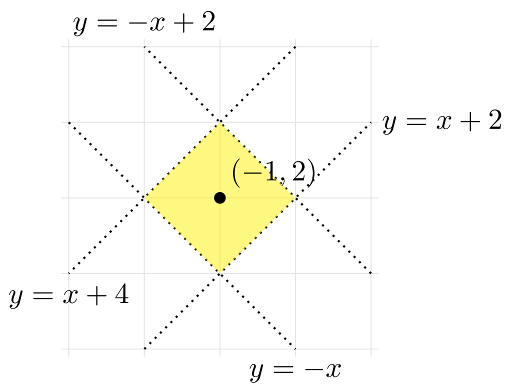
The value z_1 is positive if and only if $|x + 1| + |y - 2| < 1$. This holds if and only if

$$\pm(x + 1) \pm (y - 2) < 1.$$

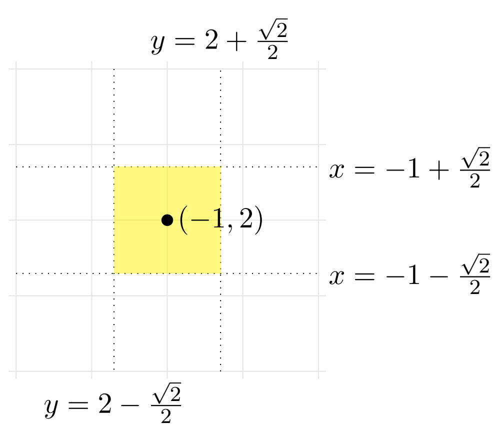
Chantalle considers the different cases:

$$\begin{aligned}
(x+1) + (y-2) &= x + y - 1 < 1 && \implies y < -x + 2,\\
(x+1) - (y-2) &= x - y + 3 < 1 && \implies y > x + 2,\\
-(x+1) + (y-2) &= -x + y - 3 < 1 && \implies y < x + 4,\\
-(x+1) - (y-2) &= -x - y + 1 < 1 && \implies y > -x.
\end{aligned}$$

The value z_1 is thus positive exactly for the following x and y values (colored yellow in the figure) and is otherwise 0 (white in the figure).



The value $z_2 = \max\{0, -2 \cdot \max\{|y - 2|, |x + 1|\} + \sqrt{2}\}$ is positive if and only if $\max\{|y - 2|, |x + 1|\} < \frac{\sqrt{2}}{2}$. These are exactly the points (x, y) for which $|y - 2|, |x + 1| < \frac{\sqrt{2}}{2}$.



Since $z = z_1 + z_2$ and neither z_1 nor z_2 can ever be negative, it follows for all points (x, y) that $z > 0$ if and only if either z_1 or z_2 (or both) are positive. Therefore, the points (x, y) where the neural network outputs a positive value are the superposition of the two yellow regions for z_1 and z_2. **This forms the following star shape, confirming that S5 is the correct sketch.**

Illustration: Zyanya Santuario

22 Discarding Cards

Author: Anne Zander (University of Twente)

Challenge

The two elves Atto and Bilbo were assigned to wrap gifts. Instead of working, they prefer to admire the presents. They have just discovered the card game “Uno” and are playing their third round. Atto is losing yet again. Frustrated, he throws the cards on the table. Then he starts pondering: “In most card games, cards have attributes, such as a color and a symbol. Suppose I have a card game with n colors and m symbols, where $n \geq 2$ and $m \geq 2$, so that each color-symbol combination appears exactly once as a card. As the sole player, I draw a given number x of random cards, where $x \geq 2$, from the entire game. Can I always arrange these x cards in such a way that they can be discarded one by one no matter which cards I picked?” By “discarding,” Atto means that—apart from the first card in the sequence—each subsequent card must either have the same color or the same symbol as the previously discarded card.
We define $x_{n,m}$ as the minimum number of drawn cards in a card game with n colors and m symbols such that it is always possible to find an order for discarding them. Which of the following statements is true?

Possible Answers

1. $x_{n,m} = (n-1)(m-1)$
2. In a card game with 4 colors and 10 symbols, one can find a selection of 30 cards such that no discarding sequence is possible.
3. $x_{n,m} = (n-2)(m-2)$
4. A discarding sequence can always be found for any selection x of drawn cards.
5. $x_{n,m} = (n-1)m - 3$
6. In a card game with 3 colors and 6 symbols, a sequence of 10 cards can always be discarded.
7. $x_{n+1,m} = x_{n,m} + m + 1$
8. $x_{n+1,m+1} = x_{n,m} + m + n - 4$
9. It holds that $x_{4,6} = x_{3,8}$
10. $x_{n,m+1} = x_{n,m} + n - 1$

Solution

The correct answer is: 10.

We present two different solution methods. The first solution is based on the exclusion principle.
In the second solution, we prove that the minimal number $x_{n,m}$ of cards in a card game with n colors and m symbols required to find a sequence where all cards can be laid down is equal to $nm - n - m + 3$. This formula is equivalent to answer option 10, as can be easily verified.

A) Exclusion Principle

We denote the colors by the numbers $1,\ 2,\ 3, \ldots$ and the symbols by $a,\ b,\ c, \ldots$. Consider the following counterexample with $n = m = 2$:

Counterexample 1
First, we note that $x_{2,2} > 2$ must hold since the cards $1a$ and $2b$ cannot be laid down directly after each other. For three cards, however, there is at least one card that shares a symbol with one other card and a color with the remaining card. Therefore, all three cards can be laid down if this card is placed as the second card. Hence, $x_{2,2} = 3$.
It follows that the answers 1, 3, 4, and 5 are incorrect.

Counterexample 2
Consider now the case $n = 3$ and $m = 2$. Thus, there are 6 cards. If answer 7 were correct, then $x_{3,2} = 3 + 2 + 1 = 6$. However, from the total of 6 cards in the game, any combination of 5 randomly selected cards can be laid down. Therefore, 6 cannot be the minimal number.
It follows that answer 7 is incorrect.

Counterexample 3
Now consider the case $n = m = 3$. If answer 8 were correct, then $x_{3,3} = 3 + 2 + 2 - 4 = 3$.
Counterexample 4
We want to exclude Answer 6. To do this, it is enough to provide an example of 10 cards that cannot be discarded. The cards $1a, 1b, \ldots, 1e,\ 2a, 2b, \ldots, 2d$, and $3f$ form such an example, because $3f$ does not match any other card in either color or symbol.
It follows that Answer 6 is incorrect.
However, if the three cards $1a$, $2b$, and $3c$ are randomly drawn from the nine cards, they cannot be laid down directly after each other.
It follows that answer 8 is incorrect.

Exclusion of Possible Answer 9
Consider a game with 24 cards where $n = 4$, $m = 6$.
First, we note that $x_{4,6} > 16$ since the card $(4, f)$ cannot be laid down in a sequence if the other cards only contain the numbers one to three and the symbols a to e. We show that 16 cards in a deck with three colors and eight symbols can always be laid down.
Case 1: If only two colors are present among these 16 cards, this is obvious.
Case 2: Consider the case where each color appears at least once among the 16 drawn cards.

1. Then there are at least four different pairs of cards with the same symbol.

2. Each color has at least one symbol that also appears in another color.

We show both through contradiction: If 1 does not hold, there are either at most one symbol appearing in all three colors or at most three symbols appearing in exactly two colors. Counting the maximum number of cards based on symbol multiplicity yields at most

$$3 + 7 \cdot 1 = 10$$

cards (for one symbol in three colors and up to seven more in at most one color) or

$$3 \cdot 2 + 5 \cdot 1 = 11$$

cards (for three symbols in two colors each and up to five more in at most one color). Both contradict the assumption of 16 cards.
If 2 does not hold, there is a color that shares no symbols with the other two colors. Let the number of cards in this color be l. The other two colors can each have at most $8 - l$ cards, resulting in at most

$$l + 2(8 - l) = 16 - l$$

cards, again fewer than the assumed 16 cards. This is also a contradiction.
From 1 and 2, it follows that there are at least two symbols appearing in two different colors each, so that all colors together can be used with at least one of these symbols. We can label the symbols as a and b. Furthermore, we can label the colors so that the cards $1a$ and $2a$ as well as $2b$ and $3b$ appear among the 16 cards. It is then easy to find a sequence in which all 16 cards can be laid down.
It follows that answer 9 is incorrect.

Exclusion of Possible Answer 2
To exclude answer option 2, we can consider the number of symbols appearing in all four colors. First, note that there must be at least 20 pairs of cards with matching symbols among the selected cards (similar to the previous argument).
Case 1: Three or more symbols appear in every color. In this case, we can simply lay down all cards of each color, using one of the mentioned symbols last to switch to a different color.
Case 2: Exactly two symbols appear in every color. Then there must be another symbol appearing in at least two colors. We can first lay down the cards of these colors, using the mentioned pair to switch from one color to another. We can then lay down all other cards of the same color and switch to another color using one of the two symbols appearing in every color.
Case 3: Exactly one symbol appears in every color. Then there are at least two more symbols appearing in one of the 20 pairs with matching symbols. Furthermore, we can even select two pairs among them that do not use exactly the same colors since otherwise, there could not be 20 pairs with matching symbols. As in the above cases, we can lay down all cards by switching between colors whenever possible and otherwise using the symbol appearing in every color.
Case 4: No symbol appears in every color. Then each symbol appears in exactly three colors. Either one color has no symbol; in this case, it is straightforward to lay down all cards. Otherwise, each color has at least one symbol appearing in two other colors. Given there are 10 symbols, at least one color must have three symbols. Among these colors, we can select the color with the most symbols, which we call color "1." Each other color must have a

symbol also appearing in color "1," as otherwise one of the remaining colors would have more symbols than color "1." Now there is at most one color sharing only one symbol with color "1," as otherwise, there could be no color "1" with three or more symbols. We can start by laying down the cards of a color sharing the fewest symbols with color "1" and then switch to color "1." We call the symbol used for switching "a." The other two colors, in addition to "a," each have at least one more symbol also appearing in color "1." One of these two colors must even have two additional symbols also appearing in "1" since otherwise, not all 10 symbols could be among the 20 cards. Thus, it is not hard to see that all cards can be laid down.

B) Proof with Graph Theory

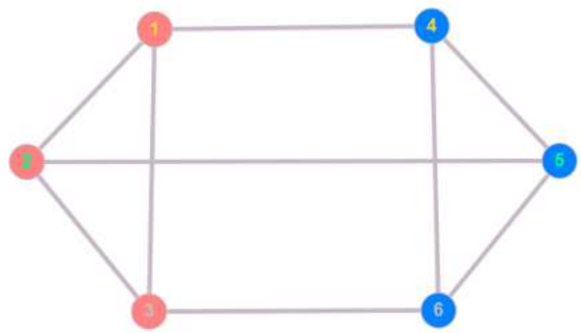


Figure 35: Card game graph for two colors and three symbols

In this proof, we represent each card in the card game as a vertex in a graph. Two vertices are connected by an edge if the corresponding cards share the same color or symbol. If we consider only x cards from the card game, this corresponds to a subgraph containing only the vertices and edges associated with the x selected cards. A valid sequence of cards becomes a **path** in the graph — a sequence of vertices connected by edges, with each vertex visited exactly once. In graph theory, such a path is called a **Hamiltonian path**.
To prove our claim, we formulate two subclaims serving as a lower and upper bound:

1. There exists a subgraph (called G_1) of the card game graph with $x = nm - n - m + 2$ vertices that does not contain a Hamiltonian path.
2. Every subgraph of the card game graph with $x = nm - n - m + 3$ vertices contains a Hamiltonian path.

As an example, Figure 35 shows the card game graph for two colors (vertices with orange and blue backgrounds) and three symbols (represented by the text colors of the vertex numbers: yellow, green, and gray).

Structure of the Card Game Graph

The card game graph contains nm vertices since each color-symbol combination appears exactly once. Furthermore, each vertex is connected to $n + m - 2$ other vertices:

- $n - 1$ vertices of the same color.
- $m - 1$ vertices of the same symbol.

Proof of Subclaim 1

To prove the first subclaim, we select any vertex, call it K, and delete all $n+m-2$ vertices connected to it. This leaves a subgraph with $nm-n-m+2$ vertices. This subgraph corresponds exactly to the graph G_1 from the claim. In this subgraph, the selected vertex K is isolated (without edges), meaning it cannot be part of a Hamiltonian path. Thus, there is no Hamiltonian path in this subgraph.

Proof of Subclaim 2

We now prove the second subclaim. All vertices corresponding to cards of the same color (or symbol) form a clique. In a clique, every pair of vertices is connected by an edge. Within such a clique, it is always possible to find a Hamiltonian path representing a direct connection between two arbitrary vertices.
Without loss of generality, we assume that $n \leq m$; otherwise, we can simply swap the roles of colors and symbols. Instead of working with the entire card game graph, we consider a reduced graph where each color clique is represented by a **meta-vertex**. A meta-vertex groups all vertices of a clique and is represented as a single vertex in the reduced graph.
Two meta-vertices are connected by an edge if at least one original vertex from each clique is connected in the original graph, i.e., if two original vertices from two different cliques share a symbol. Moreover, we will refer to an edge between two meta-vertices as strong edge if either at least one meta-vertex has only one symbol or if both meta-vertices share at least two symbols.
In the reduced graph formed from the entire deck:

- There are n meta-vertices.
- The graph is fully connected, meaning each meta-vertex is connected to every other meta-vertex via a strong edge.

We show that even after deleting $n+m-3$ original vertices, a Hamiltonian path always exists in the corresponding reduced graph.
Deleting original vertices affects the reduced graph as follows:
To delete an edge between two meta-vertices entirely, at least m original vertices (one for each symbol) must be deleted. However, it is not possible to delete all edges connected to a meta-vertex as this would require at least

- $m-1$ deletions within the shared clique,
- $n-1$ additional deletions (one in each of the other cliques).

This requires $n+m-2$ deletions, which is greater than $n+m-3$.
Since $n \leq m$, it is impossible to delete two or more edges or even strong edges in the reduced graph that don't share a meta-vertex without exceeding the deletion limit of $n+m-3$ original vertices, as $2(m-1) \geq m+n-2 > m+n-3$. This means that for two meta-vertices A and B where an edge is deleted, all other edges that get deleted must either be connected to A or B. We will show by contradiction that either all deleted edges are all connected to A or are all connected to B:

Let C be a meta-vertex such that the edge between A and C gets deleted and D be a meta-vertex such that the edge between B and D gets deleted. If C and D were different, this would contradict the fact, that the edge between A and C wouldn't share any meta-vertices with the edge from B to D. Thus, $C = D$. This also cannot be true, because this implies, that A, B and C do not share any symbols (otherwise there would be an edge between at least two of them). This however, means that the total amount of $3m$ symbols of all three meta-vertices was reduced to at most m symbols, which again exceeds the deletion limit $n + m - 3 < 2m$ under our assumption $n \leq m$. Therefore, all deleted edges are connected to the same node.

For strong edges the situation is a little different: Using the same notation for C and D as before, the argument above still implies $C = D$. On the other hand, it is now possible that between A, B and C, there a no strong edges. In this case there is a maximum of one symbol in common between each of them. Altogether, A, B and C can only have a maximum of $m + 3$ symbols, which is just within the deletion limit, but only if $n = m$ and if A, B and C are all connected by edges. This allows us to find a Hamiltonian path in the reduced graph in the following way: First we can get from A to B and then to C. From there, all other edges are strong and the remaining meta-vertices form a fully connected graph. This means, that we can use any order of the remaining vertices to obtain a Hamiltonian path in the reduced graph.
This Hamiltonian path in the reduced graph can always be extended to a Hamiltonian path in the original card game graph by mapping the Hamiltonian path from the reduced graph back to the vertices of the original graph, laying down the cards within each color clique in a valid order. That this is possible for the colors corresponding to the meta-vertices A, B, and C follows from the fact that each pair of the three meta-vertices must share exactly one distinct symbol in order to reach a total of $m + 3$ symbols. For the rest of the path this is ensured by the strong edges: If a color appears for the first time by discarding one symbol, we can discard all other symbols of that color in such a way that we can use a different symbol to get to the next color color.
The only thing left to check is the case, where all deleted edges and all edges that are not strong share the same meta-vertex A. The remaining meta-vertices again form a fully connected graph. By starting at A and using any edge to another meta-vertex we can thus find a Hamiltonian path for this case, which by the same argument as above can be used to obtain a Hamiltonian path in the original graph.

Illustration: Julia Nurit Schönnagel

23 Elf Football

Author: Dante Luber (BMS, Goethe University Frankfurt)

Task

The elves, who are diligently packing gifts for Santa, want to play football in their free time. They have repeatedly asked Santa for a football, but due to this year's tight budget, he refuses to buy one.
A clever elf named Mo, who makes gift boxes, has the brilliant idea of crafting the football himself. Mo has access to a machine that can print various body nets (2D shapes) onto sturdy paper.
However, Mo is unsure which net to print so that it can be folded and glued into a ball.
So, he asks the smart elf Mara for advice.
Mara responds: "I can't guarantee a design that will result in a perfect ball, but I know of a cool object that you can use for the game."
She shows Mo a picture of the body they can craft for the game (see Figure 36).
Excited, Mo asks Mara for the design of the pattern. Mara goes to the office and searches her computer for the file containing the net for this body. Unfortunately, she can't remember which number the design has. However, she is sure that it is among these 8 designs and prints all 8 possible nets. Mo must figure out which of the nets shown below corresponds to the object in Figure 36. Which net (1) through (8) must Mo choose to fold it into this object?

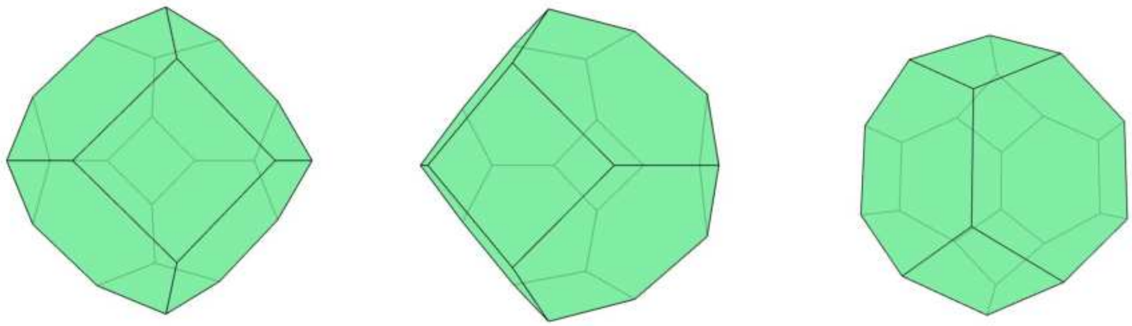


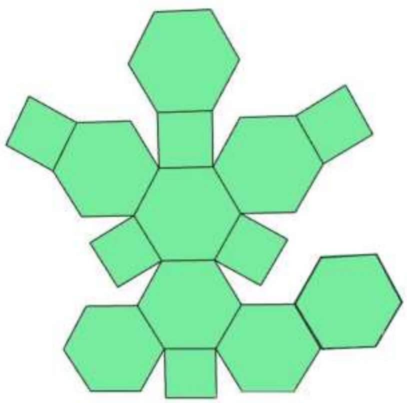
Figure 36: Three 3D images of the final football from different angles.

Possible Answers:

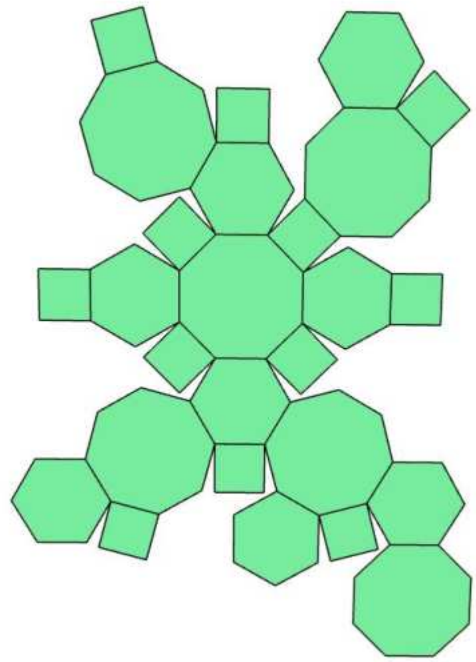
1. 1
2. 2
3. 3
4. 4
5. 5
6. 6
7. 7
8. 8
9. All
10. None

Project Reference

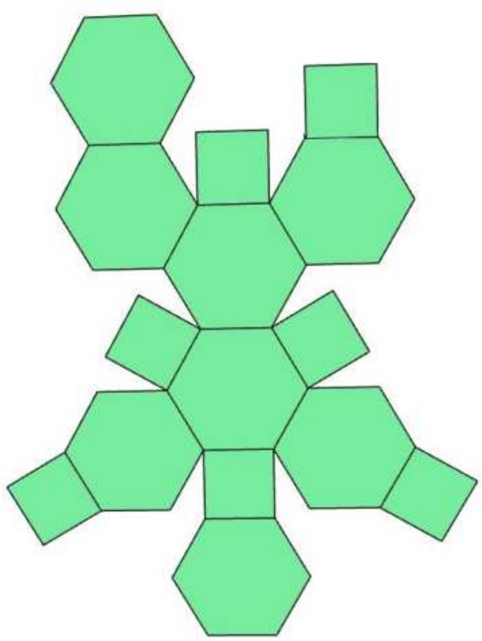
- The figures were generated using `OSCAR` — **O**pen **S**ource **C**omputer **A**lgebra **R**esearch System, Version 1.0.0, developed by The `OSCAR` Team, 2024.
- The `OSCAR` project is developed within the TRR 195 into a visionary next generation open source computer algebra system surpassing the combined mathematical capabilities of the underlying systems.
- `OSCAR` allows users to construct new mathematical objects efficiently by combining existing building blocks from any of the cornerstones and equip these objects with mathematical capabilities exceeding those of the individual systems in a transparent way.
- For more information, or if you want to use `Oscar` yourself, visit https://www.oscar-system.org.
- This challenge was inspired by MatchTheNet, a free, online, polyhedral geometry game developed by the `polymake` team. You can play the full game at https://www.matchthenet.de/.



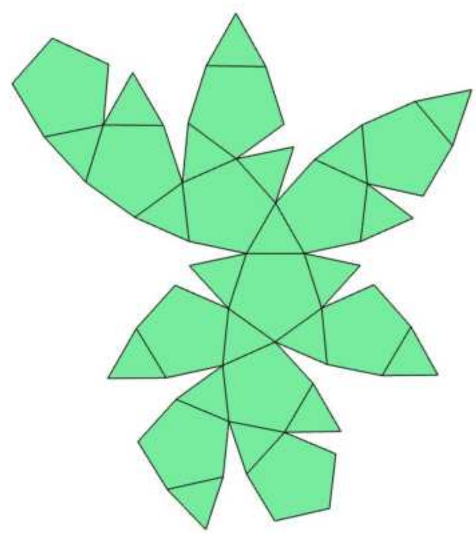
Net 1



Net 2

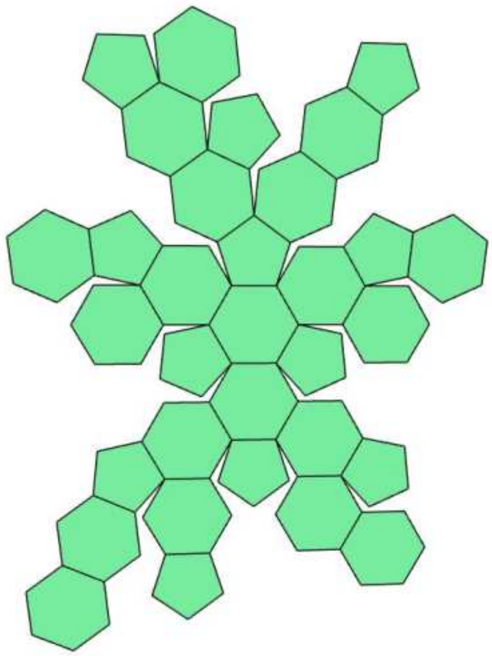


Net 3

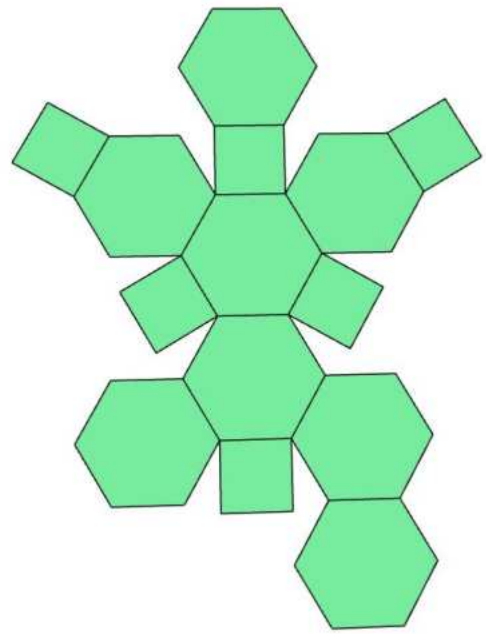


Net 4

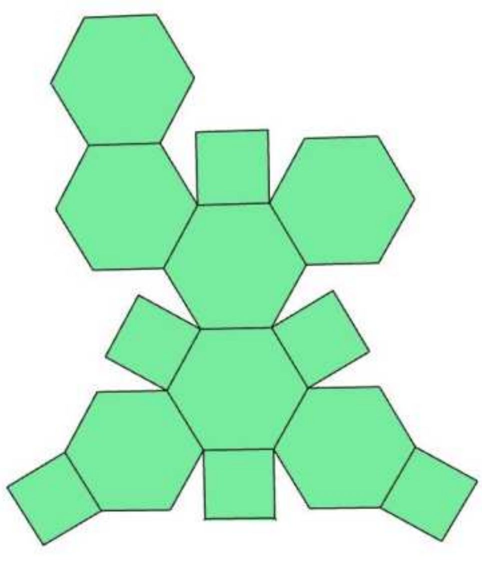
Figure 37: Possible answers (1) - (4).



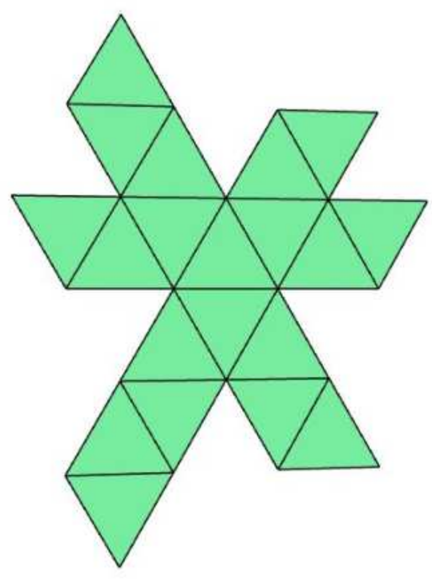
Net 5



Net 6



Net 7



Net 8

Figure 38: Possible answers (5) - (8).

Solution

The correct answer is: 6.

- *Convex polytopes* generalize squares, cubes, hexagons, tetrahedrons, and other shapes with rigid sides. The polytope in Figure 36 is the 3-dimensional *permutahedron*. The *vertices* correspond to all *permutations* (reorderings) of the vector $(1, 2, 3, 4)$.
- First, we observe that the "football" in Figure 36 consists of equally sized hexagons and squares. Hence, we exclude nets 2, 4, 5, and 8. The football has 8 hexagonal faces and 6 square faces. This eliminates nets 3 and 7.
- Consider the right hexagon in net 1. If we attach it to the adjacent square, the corners of three hexagons would meet at the corner of a square, which is not possible.

Illustration: Friederike Hofmann

24 Elves and Lebkuchenschlucker

Author: Filip Blašković (ZIB)

Challenge

On a snowy winter day, a group of busy elves traveled through the enchanted forest, their sleighs packed with sweet presents from the famous North Pole bakery. These were not just any presents — the elves had carefully prepared five delicious types of cookies: ginger, honey, cinnamon, cardamom, and nutmeg. Thousands of boxes, carefully labeled and stacked on the sleighs, were destined for Santa Claus, who needed to distribute them to children worldwide. The elves were hurrying to reach Santa's home in the valley before it was too late to begin his Christmas journey.

Just as they were about to leave the forest and enter the valley, something unexpected happened. A giant, cinnamon-colored Christmas monster named *Lebkuchenschlucker* leaped out from behind a bush, landing right in front of the elves' sleighs! With a mischievous grin, the greedy monster grabbed all the cookie boxes in one swoop. He was just about to vanish back into the depths of the forest when the elves, thinking quickly, pleaded with him.

"Please!" they cried. "Santa Claus is waiting for these gifts; he must leave soon to deliver them to all the children. Could we at least have some of the boxes back?"

Lebkuchenschlucker paused, scratching his chin. After a moment, he smiled slyly and said: "Very well, I will give you a chance. But only if you are willing to play a game."

The elves exchanged nervous glances but knew they had no choice. They nodded in agreement. The monster laid out the rules: "Here is how it will work," he said. "Each of you will receive a box of cookies, but the label on your box will be hidden. You will be able to see the labels on the other elves' boxes but not your own."

1. First, decide on a strategy.
2. Next, you will receive your presents.
3. Finally, I will call on you one by one, in an order of my choosing, and when I call on one of you, they have to guess the flavor of cookies in their box aloud, so everyone can hear it. You will all of course know which elf I have called out.

"When the game starts, you are only allowed to talk if I ask you to, and even then you're only allowed to guess your cookie taste.
At the end of the game, those who guess correctly will be allowed to keep their box and deliver it to Santa. Those who guess incorrectly must return to the bakery to get a new one."
The monster then added: "Oh, and don't think you can rely on counting the flavors — I have far more cookie boxes hidden away than there are of you! You have to rely on hearing and your memory."
The elves huddled together, deep in thought. How should they play the game to ensure they keep as many boxes as possible?

Here are the questions:

1. If there are 100 elves, what is the maximum number of cookie boxes they are guaranteed to win back using the optimal strategy?
2. What is the answer to the same question if the 100 elves play the same game, but this time they are divided into five friends cliques of 20 elves each before they get the chance to agree on a strategy? In this version, no elf within a friends clique can see the labels on the boxes of their friends during the game, but they can see the labels of all the elves in all other groups. However, all 100 of them can work on a strategy together after finding out their friends cliques, but before actually getting the cookie boxes. Does this change the outcome, and how?

Possible Answers:

1. 50 and 20
2. 50 and 40
3. 95 and 90
4. 97 and 80
5. 96 and 80
6. 96 and 96
7. 99 and 0
8. 99 and 80
9. 99 and 95
10. 100 and 100

Solution

The correct answer is: 99 and 80

First question. Obviously, the elves cannot guarantee all 100 correct answers since the elf that is called first does not have any information about the potential flavor of their box. However, the elves can guarantee 99 correct answers using the following strategy: they assign numbers $0, 1, 2, 3, 4$ to flavors before the game starts. For example: ginger - 0, honey - 1, cinnamon - 2, cardamom - 3, nutmeg - 4. Then, they guess as follows:

- *First elf.* The first elf that guesses (call him Albert) does not have any information about his box flavor so it is not guaranteed that he will guess it correctly. However, Albert's guess will serve to provide important information for the rest of the elves and help them deduce their box flavor. Albert's strategy is to sum up all the flavors (i.e. assigned numbers) of all the elves he sees – denote this sum with A – and guess the flavor $A \mod 5$ (there are 5 flavors in total). Then, the next elf (and each subsequent elf) will have sufficient information to deduce their box flavor.

- *All other elves.* Any other elf, call him Benjamin, heard Albert's answer, which gives him enough information to identify his own box flavor correctly. First, denote with B the sum of all the boxes Benjamin saw before the Lebkuchenschlucker started calling out elves, excluding Albert's box. Notice that the flavours that are included in B are exactly all the flavours that Albert sees, excluding the flavour of Benjamins cookies. Therefore, if r is the flavour of Benjamins cookies, it holds

  $$B + r = A,$$

  but as r is between zero and four, it obviously holds that $r = r \bmod 5$. But, be careful! Benjamin does not actually hear A, he hears the cookie flavour that Albert guessed, namely $A \bmod 5$. Now we have to show that

  $$A - B \equiv_5 (A \bmod 5) - B.$$

  We can write A as $5k + (A \bmod 5\ )$, for some integer k. Plug it into the above equation gives us the equivalent formulation of the question

  $$5k + (A \bmod 5\ ) - B \equiv_5 (A \bmod 5) - B.$$

  This is oviously true as $5k \equiv_5 0$.

  So, Benjamin can guess the correct cookies in his box by saying the name of the cookie flavour corresponding to the the number

  $$r \equiv_5 (A \bmod 5) - B,$$

  which is his right flavour

Second question.

- *The elves cannot guarantee more than 80 correct answers.* First, we observe that there is an order in which the Lebkuchschlucker can call the elves out, such that the first 20 have no information. Therefore, we cannot guarantee that more than 80 guess their flavour correctly. Lebkuchenschlucker can play particularly sly and call first 20 elves from the same friends clique. However, let's show that the elves can adopt a strategy that ensures 80 of them will guess correctly, regardless of the order in which they are called.

- *The elves can come up with strategy that guarantees at least 80 correct answers:* Before the game begins, they agree to divide themselves into 20 teams of five elves, with each team containing one elf from each of the five friends cliques. This setup allows each team to play independently from all other elves. Within each team, each elf can see all other team members, as never two friends are assigned to the same team. The idea is that for every team, the first person called out, takes the role of Albert, by telling the sum of all the cookie box numbers from his team. Then, if every other elf in the team acts like Benjamin, this ensures 4 correct answers per team. Since there are 20 disjoint teams, this strategy ensures that $20 \cdot 4 = 80$ correct answers are guaranteed.